Phase transitions in the Ramsey-Turán theory

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Abstract

Let $f(n)$ be a function and L be a graph. Denote by $\mathbf{RT}(n, L, f(n))$ the maximum number of edges of an L-free graph on n vertices with independence number less than $f(n)$. Erdős and Sós [1] asked if $\mathbf{RT}(n, K_5, c\sqrt{n}) = o(n^2)$ for some constant c. We answer this question by proving the stronger $\mathbf{RT}\big(n, K_5, o\big(\sqrt{n \log n}\big)\big) = o(n^2)$. It is known that $\mathbf{RT}\big(n, K_5, c\sqrt{n \log n}\big) = n^2/4 + o(n^2)$ for $c > 1$, so one can say that K_5 has a Ramsey-Turán *phase transition* at $c\sqrt{n \log n}$. We extend this result to several other K_s's and functions $f(n)$, determining many more phase transitions. We shall formulate several open problems, in particular, whether variants of the Bollobás-Erdős graph exist to give good lower bounds on $\mathbf{RT}(n, K_s, f(n))$ for various pairs of s and $f(n)$. Among others, we use Szemerédi's Regularity Lemma and the Hypergraph Dependent Random Choice Lemma. We also present a short proof of the fact that K_s-free graphs with small independence number are sparse.

Keywords: Ramsey, Turán, independence number, dependent random choice

1. Introduction

Notation. In this paper we shall consider only simple graphs, i.e., graphs without loops and multiple edges. As usual, G_n will always denote a graph on n vertices. More generally, in case of graphs the (first) subscript will always denote the number of vertices, for example K_s is the complete graph on s vertices, and $T_{n,r}$ is the r-partite Turán graph on n vertices, i.e., the complete r-partite graph on n vertices with class sizes as equal as possible. Given a graph G, we use $e(G)$ to denote its number of edges, and use $\alpha(G)$ to denote its independence number. Given a subset U of the vertex set of a graph G, we use $G[U]$ to denote the subgraph of G induced by U.

In this paper all logarithms are base 2; $\omega(n)$ denotes an arbitrary function tending to infinity slowly enough so that all calculations we use go through. Whenever we write that "$\omega(n) \to \infty$ slowly", we mean that the reader may choose an arbitrary $\omega(n) \to \infty$, the

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assertion will hold, and the more slowly $\omega(n) \to \infty$ the stronger the assertion, i.e., the theorem is. In the proofs, we shall assume that $\omega(n) = o(\log\log\log n)$. In our cases, if we prove some theorems for such functions $\omega(n)$, then these results remain valid for larger functions as well. To simplify the formulas, we shall often omit the floor and ceiling signs, assuming that they are not crucial.

Sós [2] and Erdős and Sós [1] defined the following 'Ramsey-Turán' function:

Definition 1.1. Denote by $\mathbf{RT}(n, L, m)$ the maximum number of edges of an L-free graph on n vertices with independence number less than m.

We are interested in the asymptotic behavior of $\mathbf{RT}(n, L, f(n))$ and its "phase transitions", i.e., in the question, when and how the asymptotic behavior of $\mathbf{RT}(n, L, f)$ changes sharply when we replace f by a slightly smaller g.

Definition 1.2. Let

$$\overline{\rho\tau}(L, f) = \limsup_{n\to\infty} \frac{\mathbf{RT}(n, L, f(n))}{n^2} \quad \text{and} \quad \underline{\rho\tau}(L, f) = \liminf_{n\to\infty} \frac{\mathbf{RT}(n, L, f(n))}{n^2}.$$

If $\overline{\rho\tau}(L, f) = \underline{\rho\tau}(L, f)$, then we write $\rho\tau(L, f) = \overline{\rho\tau}(L, f) = \underline{\rho\tau}(L, f)$, and call $\rho\tau$ the *Ramsey-Turán density* of L with respect to f, $\overline{\rho\tau}$ the upper, $\underline{\rho\tau}$ the lower *Ramsey-Turán densities*, respectively.

It is easy to see that $\rho\tau(L, f) = c$ is equivalent to $\mathbf{RT}(n, L, f(n)) = cn^2 + o(n^2)$. Sometimes we want to study the case when the bound on the independence number $f(n)$ is $o(g(n))$. Formally $o(g(n))$ is not a function, we shall consider $\rho\tau(L, o(g))$ as $\rho\tau(L, g/\omega)$ where $\omega(n)$ is an arbitrary function tending to infinity (slowly). More formally, let

$$\overline{\rho\tau}(L, o(g)) = \lim_{\varepsilon\to 0} \overline{\rho\tau}(L, \varepsilon g) \qquad \text{and} \qquad \underline{\rho\tau}(L, o(g)) = \lim_{\varepsilon\to 0} \underline{\rho\tau}(L, \varepsilon g).$$

Again if $\overline{\rho\tau}(L, o(g)) = \underline{\rho\tau}(L, o(g))$, then we write $\rho\tau(L, o(g)) = \overline{\rho\tau}(L, o(g)) = \underline{\rho\tau}(L, o(g))$, and in this case, we write

$$\mathbf{RT}(n, L, o(g(n))) = \rho\tau(L, o(g))n^2 + o\big(n^2\big).$$

In other words, we use $\rho\tau(L, o(g(n)))$ in the following way: if $\overline{\rho\tau}(L, f) \le c$ for every $f(n) = o(g(n))$, then $\overline{\rho\tau}(L, o(g)) \le c$. If $\underline{\rho\tau}(L, f) \ge c$ for some $f(n) = o(g(n))$, then $\underline{\rho\tau}(L, o(g)) \ge c$.

When we write $\rho\tau(L, f)$, we use f instead of $f(n)$, since $\rho\tau(L, f(n))$ would suggest that this constant depends on n. If however, we write something like $\rho\tau(L, c\sqrt{n\log n})$, that is (only) an abbreviation of $\rho\tau(L, h)$, where $h(n) = c\sqrt{n\log n}$, (see e.g. Theorem 1.4). So, even when we write $\rho\tau(L, f(n))$, we are treating $f(n)$ as a function, which means $\rho\tau(L, f(n))$ does not depend on n.[3]

The theory of $\rho\tau(L, f)$ is very complex, with many open questions. Here we focus on the case when L is a clique.

Erdős and Sós [1] determined $\mathbf{RT}(n, K_{2r+1}, o(n))$.

[3]More precisely, $f(n)$ means an "abstract" function $f : \mathbb{N} \to \mathbb{N}$, depending on n.

Theorem 1.3. *For every positive integer r,*

$$\mathbf{RT}(n, K_{2r+1}, o(n)) = \frac{1}{2}\left(1 - \frac{1}{r}\right)n^2 + o(n^2).$$

The meaning of Theorem 1.3 is that the Ramsey-Turán density of K_{2r+1} in this case is essentially the same as the Turán density $\frac{1}{2}(1-1/r)$ of K_{r+1}. In [1], Erdős and Sós proved that $\mathbf{RT}(n, K_5, c\sqrt{n}) \le n^2/8 + o(n^2)$ for every $c > 0$. [4] They also asked if $\mathbf{RT}(n, K_5, c\sqrt{n}) = o(n^2)$ for some $c > 0$. One of our main results answers this question.

Theorem 1.4.
$$\mathbf{RT}\Big(n, K_5, o\Big(\sqrt{n \log n}\Big)\Big) = o(n^2).$$

Here, by Construction 2.1, $o\big(\sqrt{n \log n}\big)$ is sharp in the sense that

$$\underline{\rho\tau}\Big(K_5, c\sqrt{n \log n}\Big) \ge \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{4} \qquad \text{for any} \qquad c > 1. \tag{1}$$

Definition 1.5 (Phase Transition)**.** Given a graph L and two functions $g(n) \le f(n)$, we shall say that L has a *phase transition* from f to g if $\overline{\rho\tau}(L, g) < \underline{\rho\tau}(L, f)$.

Given a function $\varphi(n) \to 0$, we shall say that L has a *φ-phase-transition* at f if L has a phase transition from f to φf. If L has a φ-phase-transition at f for every φ tending to 0, then we shall say that L has a *strong phase transition* at f. Let $\varphi_{\omega,\varepsilon}(n) = 2^{-\omega(n)\log^{1-\varepsilon} n}$. If there exists an $\varepsilon > 0$ for which L has a $\varphi_{\omega,\varepsilon}$-phase-transition at f for every function $\omega(n)$ tending to infinity, then we shall say that L has a *weak phase transition* at f.[5]

For example, Theorem 1.3 shows that K_5 has a strong phase transition at n, since, by Turán's Theorem, we have $\rho\tau(K_5, n) = 3/8$. Theorem 1.4 and (1) show that K_5 has a strong phase transition at $c\sqrt{n \log n}$ for every $c > 1$.

Actually, every K_s with $s > 2$ has a strong phase transition at n. More generally, given a graph G, if $\chi(G) > 2$ and G has an edge e such that $\chi(G-e) < \chi(G)$, then G has a strong phase transition at n. On the other hand, let $K_s(a_1, \ldots, a_s)$ be the complete s-partite graph with class sizes $a_1, \ldots, a_s$. Simonovits and Sós [4] showed that if $s < a \le b$, then

$$\rho\tau(K_{s+1}(a, b, \ldots, b), o(n)) = \rho\tau(K_{s+1}(a, b, \ldots, b), n) = \frac{1}{2}\left(1 - \frac{1}{s}\right),$$

which means that $K_{s+1}(a, b, \ldots, b)$ does not have a strong phase transition at $f(n) = n$.

Clearly, a strong phase transition implies a weak phase transition.

Remark: Here we investigate phase transitions in Ramsey-Turán Theory. In other words, we try to understand the following questions:

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[4]If Erdős and Sós knew the result of Ajtai, Komlós and Szemerédi [3] on the Ramsey number $\mathbf{R}(3, n)$, then they were able to prove that $\mathbf{RT}\big(n, K_5, o\big(\sqrt{n \log n}\big)\big) \le n^2/8 + o\big(n^2\big)$.

[5]The strange function $2^{\log^{1-\varepsilon} n}$ is somewhere "halfway" between $\log n$ and n^c.

(1) Given a graph L and a very large n, when do we observe crucial drops in the value of $\mathbf{RT}(n, L, m)$ when m is changing (continuously) from n to 2?
(2) For a fixed L, which functions f and g satisfy

$$\limsup_{n\to\infty} \frac{\mathbf{RT}(n, L, g(n))}{\mathbf{RT}(n, L, f(n))} < 1?$$

In this field there are several constructions providing lower bounds on Ramsey-Turán functions that are based on constructions corresponding to some "simple, small Ramsey numbers". Let $\mathbf{R}(t, m)$ be the Ramsey number: the minimum n such that every graph G_n on n vertices contains a clique K_t or an independent set of size m.

Unfortunately, we do not know Ramsey functions very well. The case $t = 3$ is well-described, we know $\mathbf{R}(3, m) = \Theta(m^2/\log m)$, see (11). For $t \geq 4$ we have only

$$\Omega\left(\frac{m^{(t+1)/2}}{(\log m)^{(t+1)/2-1/(t-2)}}\right) \leq \mathbf{R}(t, m) \leq O\left(\frac{m^{t-1}}{(\log m)^{t-2}}\right), \tag{2}$$

where the upper bound follows from Ajtai, Komlós and Szemerédi [3] and the lower bound follows from Bohman and Keevash [5]. It is conjectured that the upper bound is sharp up to some $\log m$-power factors.

We define the 'inverse' function $\mathbf{Q}(t, n)$ of $\mathbf{R}(t, m)$, i.e., the minimum independence number of K_t-free graphs on n vertices. It is an inverse function in the sense that if $\mathbf{R}(t, m) = n$, then $\mathbf{Q}(t, n) = m$. For example, $\mathbf{Q}(2, n) = n, \mathbf{Q}(3, n) = \Theta(\sqrt{n \log n})$ and $\Omega(n^{1/3} \log^{2/3} n) \leq \mathbf{Q}(4, n) = O(n^{2/5} \log^{4/5} n)$. In general, for $t \geq 3$, we know from (2) that

$$\Omega\left(n^{\frac{1}{t-1}} (\log n)^{\frac{t-2}{t-1}}\right) \leq \mathbf{Q}(t, n) \leq O\left(n^{\frac{2}{t+1}} (\log n)^{1-\frac{2}{(t-2)(t+1)}}\right). \tag{3}$$

We study $\mathbf{RT}(n, K_s, \mathbf{Q}(t, f(n)))$ for various functions f to find phase transitions. Ramsey-Turán problems with independence number $\mathbf{Q}(t, f(n))$ were also studied earlier in a somewhat different way. Given an integer $d \geq 2$, define the *d-independence number* $\alpha_d(G)$ of G to be the maximum size of a vertex set S for which $G[S]$ contains no K_d. For example, the independence number $\alpha(G)$ of G is $\alpha_2(G)$. Denote by $\mathbf{RT_d}(n, L, f(n))$ the maximum number of edges of an L-free graph on n vertices with d-independence number less than $f(n)$. It is easy to see that $\alpha(G_n) < \mathbf{Q}(d, f(n))$ implies $\alpha_d(G_n) < f(n)$, so $\mathbf{RT}(n, L, \mathbf{Q}(d, f(n))) \leq \mathbf{RT_d}(n, L, f(n))$. Therefore an upper bound on $\mathbf{RT_d}(n, L, f(n))$ is also an upper bound on $\mathbf{RT}(n, L, \mathbf{Q}(d, f(n)))$. Erdős, Hajnal, Simonovits, Sós and Szemerédi [6] gave an upper bound on $\mathbf{RT_d}(n, K_s, o(n))$, implying the following theorem.

Theorem 1.6. *For any function $\omega(n)$ tending to infinity, if $2 \leq t < s$, then*

$$\overline{\rho\tau}\left(K_s, \mathbf{Q}\left(t, \frac{n}{\omega(n)}\right)\right) \leq \frac{s-t-1}{2s-2}. \tag{4}$$

Lower bounds on $\mathbf{RT_d}(n, K_s, o(n))$ were provided by constructions of Balogh and Lenz [7, 8]. Unfortunately, a lower bound on $\mathbf{RT_d}(n, L, f(n))$ provides no lower bound on $\mathbf{RT}(n, L,$

$\mathbf{Q}(d, f(n)))$. For example, Balogh and Lenz [8] gave a construction showing that $\mathbf{RT}_3(n, K_5, f(n)) \geq n^2/16 + o(n^2)$ for some $f(n) = o(n)$; on the other hand, Theorem 1.4 implies that $\mathbf{RT}(n, K_5, \mathbf{Q}(3, f(n))) = o(n^2)$ for any $f(n) = o(n)$.

Note that $\overline{\rho\tau}(K_s, \mathbf{Q}(t,n)/\omega(n)) \leq \overline{\rho\tau}(K_s, \mathbf{Q}(t, n/\omega(n)))$, so Theorem 1.6 gives an upper bound on $\overline{\rho\tau}(K_s, \mathbf{Q}(t,n)/\omega(n))$. Construction 2.1 provides K_s-free graphs with many edges and small independence number, giving a lower bound on $\underline{\rho\tau}(K_s, \mathbf{Q}(t,n))$. Using these two results, we give conditions on s and t under which $\overline{\rho\tau}(K_s, \mathbf{Q}(t,n)/\omega(n)) < \underline{\rho\tau}(K_s, \mathbf{Q}(t,n))$ for any $\omega(n)$ tending to infinity, i.e., K_s has a strong phase transition at $\mathbf{Q}(t,n)$.

Theorem 1.7. *If $s-1 = r(t-1) + \ell$ with $0 \leq \ell < t-1, \ell < r$ and $2 \leq t < s$, then K_s has a strong phase transition at $f(n) = \mathbf{Q}(t,n)$.*

We have seen that K_5 has a strong phase transition at $c\sqrt{n \log n}$ for any $c > 1$. It follows from Theorem 1.7 that every clique K_s with $s \geq 5$ has a phase transition at $\mathbf{Q}(3,n) = \Theta\big(\sqrt{n \log n}\big)$. On the other hand, $s = 9$ and $t = 4$ do not satisfy the condition of Theorem 1.7, and we can see from Table 1 that K_9 does not have a strong phase transition at $\mathbf{Q}(4,n)$. Theorem 1.7 also implies that for any integer $K > 0$, there exists an s such that K_s has more than K strong phase transitions. For example, if $s = K! + 1$, then K_s has a strong phase transition at $\mathbf{Q}(t,n)$ for every t between 2 and $K+1$.

We also study weak phase transitions.

Theorem 1.8. *If K_s has a phase transition from $\mathbf{Q}(t,n)$ to $\mathbf{Q}(t+1,n)$, then K_s has a weak phase transition at $\mathbf{Q}(t,n)$.*

We would like to have a similar result for strong phase transitions. Unfortunately, we can prove it only by assuming some conditions on Ramsey numbers. There are some famous conjectures on $\mathbf{R}(\ell, n)$:

Conjecture 1.9 (Folklore)**.** *For every integer $\ell \geq 3$, $\mathbf{R}(\ell-1, n) = o(\mathbf{R}(\ell, n))$ as $n \to \infty$.*

This would immediately follow from the following stronger conjecture.

Conjecture 1.10. *For every integer $\ell \geq 3$, there exist $\vartheta = \vartheta(\ell) > 0$ and $N = N(\ell) > 0$ such that if $n > N$, then*

$$\mathbf{R}(\ell-1, n) \leq \mathbf{R}(\ell, n)/n^{\vartheta}. \tag{5}$$

Actually, we can formulate a much stronger Conjecture.

Conjecture 1.11. *For some constant $\gamma = \gamma(t)$,*

$$\mathbf{Q}(t,n) \approx \sqrt[t-1]{n} \log^{\gamma} n,$$

or at least

$$\sqrt[t-1]{n} < \mathbf{Q}(t,n) < \sqrt[t-1]{n} \log^{\gamma} n.$$

Many of our results depend on Conjecture 1.10 and analogous conjectures. For example, (3) and (5) imply that there exists a ϑ' such that

$$\mathbf{R}(\ell-1, \mathbf{Q}(\ell,n)) \le n^{1-\vartheta'}.$$

We know that Conjecture 1.10 is true for $\ell = 3, 4$, but for larger ℓ's we are very far from proving what is conjectured or would be needed for our purposes.

If Conjecture 1.10 is true for $\ell = t$, then we can determine $\mathbf{RT}(n, K_s, \mathbf{Q}(t,n))$. Our next result is an analogue of Theorem 1.3.

Theorem 1.12. *If $r = \lfloor \frac{s-1}{t-1} \rfloor$ and Conjecture 1.10 is true for $\ell = t$, then*

$$\rho\tau(K_s, \mathbf{Q}(t,n)) = \frac{1}{2}\left(1 - \frac{1}{r}\right).$$

We also prove an extension of Theorem 1.8.

Theorem 1.13. *If $t \ge 2$, Conjecture 1.10 is true for $\ell = t$ and $t+1$, and K_s has a phase transition from $\mathbf{Q}(t,n)$ to $\mathbf{Q}(t+1,n)$, then K_s has a strong phase transition at $\mathbf{Q}(t,n)$.*

If Conjecture 1.11 is true, then what Theorem 1.13 says is that if there is a drop on the Ramsey-Turán density while the independence number go down from $n^{\frac{1}{t}+o(1)}$ to $n^{\frac{1}{t+1}+o(1)}$, then there is a drop around $n^{\frac{1}{t}+o(1)}$.

We also characterize weak phase transitions for cliques.

Theorem 1.14. *If Conjecture 1.10 is true for $\ell = t+1$ and K_s has a phase transition from $\mathbf{Q}(t,n)$ to $\mathbf{Q}(t+1,n)$, then there exists an $\varepsilon > 0$ such that for every $\omega(n) \to \infty$ slowly, if $\varphi_\varepsilon(n) = 2^{-\omega(n)\log^{1-\varepsilon} n}$, then K_s has a φ_ε-phase-transition, i.e., weak phase transition at $\mathbf{Q}(t,n)$, and K_s does not have a phase transition from $\varphi_\varepsilon(n)\mathbf{Q}(t,n)$ to $\mathbf{Q}(t+1,n)$.*

The rest of the paper is organized as follows. In Section 2 we provide additional history of Ramsey-Turán type problems. Our aim in general is to determine the phase transitions for cliques. We summarize our results for small cliques in Section 3, see Table 1, and we state them in general in Section 4. In Section 5 we provide the main tools for our proofs: the Dependent Random Choice Lemma and the Hypergraph Dependent Random Choice Lemma. We prove our main results in Sections 6 and 7, and Section 8 contains some concluding remarks and open problems.

Remark: Independently of our work, about the same time, Fox, Loh and Zhao [9] studied the critical window of the phase transitions of K_4. Among others, they computed the dependency of the constants of the Bollobás-Erdős graph, and by introducing a new twist on the Dependent Random Choice Lemma, they substantially improved the lower bound on the independence number at the critical point.

2. History

Let $H_{k,\ell}$ denote a "Ramsey" graph on k vertices not containing K_ℓ, having the minimum possible independence number under this condition. The graph $H_{k,\ell}$ is sparse, i.e., it has $o(k^2)$ edges, see Theorem 8.1. For Theorem 1.3, Erdős and Sós [1] used $H_{n/r,3}$ to construct a graph S_n to provide the lower bound on $\rho\tau(K_{2r+1}, o(n))$. Their idea was that when a Ramsey-graph $H_{n/r,3}$ is placed into each class of a Turán graph $T_{n,r}$, we get a K_{2r+1}-free graph sequence $\{S_n\}$ with

$$e(S_n) \approx e(T_{n,r}) \qquad \text{and} \qquad \alpha(S_n) = \alpha(H_{n/r,3}) = o(n). \tag{6}$$

It is trivial to generalize this idea to give a lower bound on $\rho\tau(K_{rs+1}, o(n))$.

Construction 2.1 (Extended/Modified Erdős-Sós Construction)**.**
Let $k = \lfloor n/r \rfloor$, take a Turán graph $T_{n,r}$ with r classes and place an $H_{k,t+1}$ into each of its classes.

It is easy to see that this graph is K_{rt+1}-free, hence

$$\underline{\rho\tau}(K_{rt+1}, \alpha(H_{n/r,t+1})) \geq \frac{1}{2}\left(1 - \frac{1}{r}\right). \tag{7}$$

If Conjecture 1.10 is true for $\ell = t+1$, then $\rho\tau(K_{rt+1}, \alpha(H_{n/r,t+1}))$ exists and (7) is sharp, see Corollary 4.7.

Szemerédi [10], using his regularity lemma [11], proved $\overline{\rho\tau}(K_4, o(n)) \leq 1/8$. Bollobás and Erdős [12] constructed the so-called Bollobás-Erdős graph, one of the most important constructions in this area, which shows that $\underline{\rho\tau}(K_4, o(n)) \geq 1/8$. Indeed, the Bollobás-Erdős graph on n vertices is K_4-free, with $(\frac{1}{8}+o(1))n^2$ edges and independence number $o(n)$. Later, Erdős, Hajnal, Sós and Szemerédi [13] extended these results, determining $\mathbf{RT}(n, K_{2r}, o(n))$:

Theorem 2.2.

$$\mathbf{RT}(n, K_{2r}, o(n)) = \frac{3r-5}{6r-4}n^2 + o(n^2).$$

The lower bound is provided by their generalization of the Bollobás-Erdős graph:

Construction 2.3. Fix $h = \left\lfloor \frac{4n}{3r-2} \right\rfloor$ and $k = \left\lfloor \frac{3n}{3r-2} \right\rfloor$. Let B_h be a Bollobás-Erdős graph on h vertices. We take a B_h and a Turán graph $T_{n-h,r-2}$, join each vertex of B_h to each vertex of $T_{n-h,r-2}$, and place an $H_{k,3}$ into each class of $T_{n-h,r-2}$.

Here h was chosen to maximize the number of edges, which is equivalent with making the degrees (almost) equal. It is easy to see that this graph is K_{2r}-free. Since $\alpha(B_h) = o(n)$ and $\alpha(H_{k,3}) = o(n)$, it gives the lower bound of Theorem 2.2.

In the last years, many important, new results were proved on $\rho\tau(K_4, o(n))$. Let

$$f(n) = n2^{-\omega(n)\sqrt{\log n}} \qquad \text{and} \qquad g(n) = n2^{-o\left(\sqrt{\frac{\log n}{\log\log n}}\right)}. \tag{8}$$

Sudakov [14] proved that $\rho\tau(K_4, f) = 0$. Recently, by finding good quantitative estimates for the relevant parameters of the Bollobás-Erdős graph, Fox, Loh and Zhao [9] proved that $\underline{\rho\tau}(K_4, g) \geq 1/8$ for g of (8), complementing Sudakov's result.

Recall that $\mathbf{Q}(t, n)$ is the minimum independence number of K_t-free graphs on n vertices. So we have $\mathbf{Q}(t+1, n/r) = \alpha(H_{n/r,t+1})$, and we can write (7) as

$$\underline{\rho\tau}\Big(K_{rt+1}, \mathbf{Q}\Big(t+1, \frac{n}{r}\Big)\Big) \geq \frac{1}{2}\Big(1 - \frac{1}{r}\Big) = \frac{r-1}{2r}. \tag{9}$$

In particular, we get the following sharpening of the lower bound of Theorem 1.3:

$$\underline{\rho\tau}\Big(K_{2r+1}, \mathbf{Q}\Big(3, \frac{n}{r}\Big)\Big) \geq \frac{1}{2}\Big(1 - \frac{1}{r}\Big) = \frac{r-1}{2r}. \tag{10}$$

Though $\mathbf{Q}(3, n) = o(n)$ was sufficient for Theorem 1.3, it is not enough for our purposes. Fortunately, there exist much more accurate quantitative estimates of $\mathbf{Q}(3, n)$, so we can get more accurate information on $\rho\tau(K_{2r+1}, o(n))$. The bound $\mathbf{Q}(3, n) = \Theta\big(\sqrt{n \log n}\big)$ was proved by Ajtai, Komlós, Szemerédi [3] and Kim [15]. The best known quantitative estimates are proved by Shearer [16], Pontiveros, Griffiths, Morris, Oliveira [17] and Bohman, Keevash [18]. The bounds are

$$(1/4 - o(1))m^2/\log m \leq \quad \mathbf{R}(3, m) \quad \leq (1 + o(1))m^2/\log m \qquad \text{and} \tag{11}$$

$$\Big(1\Big/\sqrt{2} - o(1)\Big)\sqrt{n \log n} \leq \quad \mathbf{Q}(3, n) \quad \leq \Big(\sqrt{2} + o(1)\Big)\sqrt{n \log n}. \tag{12}$$

Combining (10) and (12), we have the following relation.[6] For any $c > 1$,

$$\underline{\rho\tau}\Big(K_6, c\sqrt{n \log n}\Big) \geq \underline{\rho\tau}\Big(K_5, c\sqrt{n \log n}\Big) \geq 1/4. \tag{13}$$

3. Phase transitions for cliques

If Conjecture 1.10 is true, then the assumptions of all Theorems and Corollaries in Section 4 also hold. In this section, we assume Conjecture 1.10 is true and study phase transitions of cliques. We summarize our results in Section 4 by listing $\rho\tau(K_s, f)$ for $s \leq 13$ in Table 1, which makes our results easier to understand. The first row $f(n) = n$ is Turán's theorem, the second row $o(n)$ is Theorem 1.3 if s is odd, and Theorem 2.2 if s is even. In general we have three types of functions $f(n)$:

1. $\mathbf{Q}(t, n)$: Bounds are obtained by Construction 2.1.
2. $\mathbf{Q}\Big(t, \frac{n}{\omega(n)}\Big)$:
   - $s = 2t - 1$: Bounds are obtained by Theorem 4.2.
   - $s = 2t$: Bounds are obtained by Corollary 4.10.

[6] Essentially this appears in Erdős-Sós [1].

- t divides $s-1$: Bounds are obtained by Corollary 4.11.
- other cases: Bounds are obtained by Theorem 1.6.

3. $\mathbf{Q}(t, g_q(n))$ where $g_q(n) = n2^{-\omega(n)\log^{1-1/q} n}$:
    - t does not divide s: Bounds are obtained by Corollary 4.4.
    - $s = 2t$: Bounds are obtained by Theorem 4.5.
    - $s = qt, q \geq 3$: Bounds are obtained by Corollary 4.7.

Note that under the assumption that Conjecture 1.10 is true, our results for $f(n) = \mathbf{Q}\left(t, \frac{n}{\omega(n)}\right)$ can be viewed as results on $\rho\tau(K_s, o(\mathbf{Q}(t,n)))$. Conjecture 1.10 is true for $\ell = 3, 4$, therefore, our results on $K_4, \ldots, K_8$ and results in Row 1 to Row 7 do not depend on Conjecture 1.10.

An entry "λ" in the row $f(n)$ and the column K_s means $\rho\tau(K_s, f) = \lambda$, and "$\leq \lambda$" means $\overline{\rho\tau}(K_s, f) \leq \lambda$. In the row $\mathbf{Q}(t, g_q(n))$, the entries are "$q_0 : \lambda$" meaning that $\rho\tau(K_s, \mathbf{Q}(t, g_{q_0})) = \lambda$.

Unfortunately, we know only a few equalities for the case $f(n) = \mathbf{Q}\left(t, \frac{n}{\omega(n)}\right)$. The inequality results, especially the inequalities obtained by Theorem 1.6, are unlikely to be sharp. In some small cases we could improve the upper bounds, but we do not feel that we captured the truth and we shall discuss these partial results somewhere else.

The most interesting case is K_6. A construction improving the lower bound on $\underline{\rho\tau}\big(K_6, o\big(\sqrt{n \log n}\big)\big)$ would imply several improvements in the spirit of Construction 2.3, i.e., then we would replace the Bollobás-Erdős graph in Construction 2.3 with this new construction for K_6, replace $H_{k,3}$ with $H_{k,4}$, and then optimize the class sizes.

Example: We list the details for K_{13} here. Let $\omega(n)$ be any function tending to infinity. The extremal number of K_{13} with no restriction on the independence number is $11n^2/24$, realized by the complete 12-partite Turán graph. Below we shall use $\mathbf{Q}(t,n) \overset{RC}{\sim} \sqrt[t-1]{n}$ to indicate that we assume Conjecture 1.11. When the independence number $f(n)$ is restricted by $f(n) = o(n)$ or more precisely to $\mathbf{Q}(3,n)$, we have that $\rho\tau(K_{13}, f)$ drops to $5/12$, realized by Construction 2.1. When $f(n)$ is between $\mathbf{Q}(4,n) \overset{RC}{\sim} \sqrt[3]{n}$ and $\mathbf{Q}\left(3, \frac{n}{\omega(n)}\right) \sim \sqrt{n}$, Corollaries 4.4 and 4.11 show that $\rho\tau(K_{13}, f) = 3/8$. When $f(n)$ is between $\mathbf{Q}(5,n) \overset{RC}{\sim} \sqrt[4]{n}$ and $\mathbf{Q}\left(4, \frac{n}{\omega(n)}\right) \overset{RC}{\sim} \sqrt[3]{n}$, Corollaries 4.4 and 4.11 show that $\rho\tau(K_{13}, f) = 1/3$. When $f(n)$ drops to $\mathbf{Q}\left(5, \frac{n}{\omega(n)}\right) \overset{RC}{\sim} \sqrt[4]{n}$, Theorem 1.6 shows that $\overline{\rho\tau}(K_{13}, f) \leq 7/24$. When $f(n)$ is between $\mathbf{Q}(7,n) \overset{RC}{\sim} \sqrt[6]{n}$ and $\mathbf{Q}(5, n2^{-\omega(n)\sqrt{\log n}}) \overset{RC}{\sim} \sqrt[4]{n}$, Corollary 4.4 yields that $\rho\tau(K_{13}, f) = 1/4$. Finally, when the independence number $f(n)$ is restricted by $f(n) \leq \mathbf{Q}\left(7, \frac{n}{\omega(n)}\right)$, Theorem 4.2 implies that $\rho\tau(K_{13}, f)$ drops to 0.

4. General Results

First we state one of our main results, Theorem 1.4, in a sharper form.

		K_4	K_5	K_6	K_7	K_8	K_9	K_{10}	K_{11}	K_{12}	K_{13}
1	n	$\frac{1}{3}$	$\frac{3}{8}$	$\frac{2}{5}$	$\frac{5}{12}$	$\frac{3}{7}$	$\frac{7}{16}$	$\frac{4}{9}$	$\frac{9}{20}$	$\frac{5}{11}$	$\frac{11}{24}$
2	$o(n)$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{2}{7}$	$\frac{1}{3}$	$\frac{7}{20}$	$\frac{3}{8}$	$\frac{5}{13}$	$\frac{2}{5}$	$\frac{13}{32}$	$\frac{5}{12}$
3	$g_q(n)$	$2:0$		$3:\frac{1}{4}$		$4:\frac{1}{3}$		$5:\frac{3}{8}$		$6:\frac{2}{5}$	
4	$\mathbf{Q}(3,n)$		$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{5}{12}$
5	$o\left(\sqrt{n\log n}\right)$		0	$\leq\frac{1}{6}$	$\frac{1}{4}$	$\leq\frac{2}{7}$	$\leq\frac{5}{16}$	$\frac{1}{3}$	$\leq\frac{7}{20}$	$\leq\frac{8}{22}$	$\frac{3}{8}$
6	$\mathbf{Q}(3,g_q(n))$			$2:0$		$2:\frac{1}{4}$	$3:\frac{1}{4}$		$3:\frac{1}{3}$	$4:\frac{1}{3}$	
7	$\mathbf{Q}(4,n)$				$\frac{1}{4}$	$\frac{1}{4}$		$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{3}{8}$
8	$\mathbf{Q}\left(4,\frac{n}{\omega(n)}\right)$				0	$\leq\frac{3}{16}$		$\leq\frac{5}{18}$	$\leq\frac{3}{10}$	$\leq\frac{7}{22}$	$\frac{1}{3}$
9	$\mathbf{Q}(4,g_q(n))$					$2:0$		$2:\frac{1}{4}$	$2:\frac{1}{4}$	$3:\frac{1}{4}$	
10	$\mathbf{Q}(5,n)$						$\frac{1}{4}$	$\frac{1}{4}$			$\frac{1}{3}$
11	$\mathbf{Q}\left(5,\frac{n}{\omega(n)}\right)$						0	$\leq\frac{1}{5}$			$\leq\frac{7}{24}$
12	$\mathbf{Q}(5,g_q(n))$							$2:0$			$2:\frac{1}{4}$
13	$\mathbf{Q}(6,n)$								$\frac{1}{4}$	$\frac{1}{4}$	
14	$\mathbf{Q}\left(6,\frac{n}{\omega(n)}\right)$								0	$\leq\frac{5}{24}$	
15	$\mathbf{Q}(6,g_q(n))$									$2:0$	
16	$\mathbf{Q}(7,n)$										$\frac{1}{4}$
17	$\mathbf{Q}\left(7,\frac{n}{\omega(n)}\right)$										0

Table 1: Phase Transitions.

Theorem 4.1. *If $\omega(n) \to \infty$, then*

$$\mathbf{RT}\left(n, K_5, \frac{\sqrt{n \log n}}{\omega(n)}\right) \le \frac{n^2}{\sqrt[4]{\omega(n)}} = o\big(n^2\big). \tag{14}$$

We use $\mathbf{Q}(t, n/\omega(n))$ as a bound on the independence number to generalize Theorem 4.1.

Theorem 4.2. *Suppose $p \ge 3$. If $\omega(n) \to \infty$ and there exists a constant $\vartheta > 0$ such that for every n sufficiently large, we have*

$$\mathbf{R}\left(p-1, \mathbf{Q}\left(p, \frac{n}{\omega(n)}\right)\right) < n^{1-\vartheta}, \tag{15}$$

$$\textit{then} \qquad \mathbf{RT}\left(n, K_{2p-1}, \mathbf{Q}\left(p, \frac{n}{\omega(n)}\right)\right) \le \frac{n^2}{\omega(n)^{\frac{\vartheta}{2p}}} = o\big(n^2\big).$$

Remark: It is known that condition (15) of Theorem 4.2 is satisfied for $p = 3, 4$ as $\mathbf{R}(2, \mathbf{Q}(3, n/\omega(n))) = o\big(\sqrt{n \log n}\big)$ and $\mathbf{R}(3, \mathbf{Q}(4, n/\omega(n))) = o(n^{4/5} \log^{3/5} n)$. For a p satisfying condition (15), Theorem 4.2 is best possible in the sense that $\underline{\rho\tau}(K_{2p-1}, \mathbf{Q}(p, n/2)) \ge 1/4$, as we know from (9).

We generalize Theorem 4.2 from cliques of odd size to many other sizes.

Theorem 4.3. *Suppose $p \ge 3$ and $q \ge 2$. Let $\omega(n) \to \infty$ and $f(n) = n2^{-\omega(n) \log^{\frac{q-2}{q-1}} n}$. If there exists a constant $\vartheta > 0$ such that for every n sufficiently large, we have $\mathbf{R}(p-1, \mathbf{Q}(p, f(n))) < n^{1-\vartheta}$, then*

$$\overline{\rho\tau}(K_{pq-1}, \mathbf{Q}(p, f)) \le \frac{q-2}{2q-2}.$$

Remark: Substituting $q = 2$ into Theorem 4.3, we get Theorem 4.2.

Corollary 4.4. *Suppose $p \ge 3$ and $q \ge 2$. Let $\omega(n) \to \infty$ and $f(n)$ be as in Theorem 4.3. For $1 \le i \le p-1$, let $t = \left\lfloor \frac{pq-i-1}{q-1} \right\rfloor \ge p$. If there exists a constant $\vartheta > 0$ such that for every n sufficiently large, we have*

$$\mathbf{R}(p-1, \mathbf{Q}(p, f(n))) < n^{1-\vartheta}, \tag{16}$$

then for

$$\mathbf{Q}\left(t+1, \frac{n}{q-1}\right) < g(n) < \mathbf{Q}(p, f(n)), \tag{17}$$

we have

$$\rho\tau(K_{pq-i}, g) = \frac{q-2}{2q-2}.$$

Proof of Corollary 4.4. The upper bound is obvious from Theorem 4.3. Let $r = q-1$, then $rt+1 \le pq-i$, so the lower bound is realized by (9) with the parameters r, t as above:

$$\underline{\rho\tau}\left(K_{pq-i}, \mathbf{Q}\left(t+1, \frac{n}{q-1}\right)\right) \ge \underline{\rho\tau}\left(K_{rt+1}, \mathbf{Q}\left(t+1, \frac{n}{r}\right)\right) \ge \frac{q-2}{2q-2}. \qquad \blacksquare$$

Similarly to Theorem 4.2, condition (16) of Corollary 4.4 probably is satisfied for every p, but this is not known. We know it for $p = 3, 4$. The left end of the interval given by (17) is likely smaller than the right end for every p, but again it is not known. We know it for $p = 2, 3$.

Recall that Sudakov proved that $\rho\tau\left(K_4, n2^{-\omega(n)\sqrt{\log n}}\right) = 0$, which is a special case of his more general theorem [14]:

Theorem 4.5 (Sudakov). *Let $t \ge 2$ and $\omega(n) \to \infty$. If $g(n) = \mathbf{Q}\left(t, n2^{-\omega(n)\sqrt{\log n}}\right)$, then $\rho\tau(K_{2t}, g) = 0$.*

We extend Theorem 4.5 from K_{2t} to K_{pq}, with several other functions $g(n)$. Theorem 4.6 can be compared to Theorem 4.3, where similar statement was proved for K_{pq-1} and a slightly larger $f(n)$.

Theorem 4.6. *Suppose $p \ge 2$ and $q \ge 2$. Let $\omega(n) \to \infty$ and*

$$f(n) = n2^{-\omega(n)\log^{1-1/q} n}. \quad \textit{Then} \quad \overline{\rho\tau}(K_{pq}, \mathbf{Q}(p, f)) \le \frac{q-2}{2q-2}.$$

Remark: Applying Theorem 4.6 with $p = t$ and $q = 2$, we obtain Theorem 4.5.

Corollary 4.7. *Suppose $p \ge 2$ and $q \ge 2$. Let $\omega(n) \to \infty$ and $f(n)$ be as in Theorem 4.6. For $0 \le i \le p-1$, let $t = \left\lfloor \frac{pq-i-1}{q-1} \right\rfloor \ge p$. If*

$$\mathbf{Q}\left(t+1, \frac{n}{q-1}\right) \le g(n) \le \mathbf{Q}(p, f(n)), \quad \textit{then} \quad \rho\tau(K_{pq-i}, g) = \frac{q-2}{2q-2}.$$

Proof of Corollary 4.7. We apply Theorem 4.6 to get the upper bound. Let $r = q-1$, which implies $rt+1 \le pq-i$, so the lower bound is realized by (9) with the parameters r, t as above:

$$\underline{\rho\tau}\left(K_{pq-i}, \mathbf{Q}\left(t+1, \frac{n}{r}\right)\right) \ge \underline{\rho\tau}\left(K_{rt+1}, \mathbf{Q}\left(t+1, \frac{n}{r}\right)\right) \ge \frac{q-2}{2q-2}. \qquad \blacksquare$$

Theorem 4.2 determines $\rho\tau\left(K_{2t-1}, \mathbf{Q}\left(t, \frac{n}{\omega(n)}\right)\right)$. Now we consider the even-size clique case $\rho\tau\left(K_{2t}, \mathbf{Q}\left(t, \frac{n}{\omega(n)}\right)\right)$. This case was studied by Erdős, Hajnal, Simonovits, Sós and Szemerédi [6]. They proved an upper bound for $t = 3$.

Theorem 4.8. *If $\omega(n) \to \infty$, then*

$$\mathbf{RT}\left(n, K_6, \mathbf{Q}\left(3, \frac{n}{\omega(n)}\right)\right) \leq \frac{n^2}{6} + o(n^2).$$

Using similar methods, one can prove the following general result.

Theorem 4.9. *Let $\omega(n) \to \infty$ and $f(n) = n2^{-\omega(n)\log^{1-1/q} n}$. If $2t \leq pq$ and $\mathbf{Q}\left(t, \frac{n}{\omega(n)}\right) \leq \mathbf{Q}(p, f(n))$, then*

$$\overline{\rho\tau}\left(K_{2t}, \mathbf{Q}\left(t, \frac{n}{\omega(n)}\right)\right) \leq \frac{(t-1)(q-2)}{2t(q-1)}.$$

Applying this with $t = 3$, $p = 2$ and $q = 3$, then we obtain Theorem 4.8. This is a special case of the following Corollary:

Corollary 4.10. *For $t \geq 3$, if $\mathbf{Q}\left(t, \frac{n}{\omega(n)}\right) \leq \mathbf{Q}\left(t-1, n2^{-\omega(n)\log^{2/3} n}\right)$, then*

$$\overline{\rho\tau}\left(K_{2t}, \mathbf{Q}\left(t, \frac{n}{\omega(n)}\right)\right) \leq \frac{t-1}{4t}.$$

Proof. The condition $t \geq 3$ implies $2t \leq 3(t-1)$, therefore we can apply Theorem 4.9 with $p = t-1$ and $q = 3$. ∎

Remark: It is known that the condition of Corollary 4.10 is satisfied for $t = 3, 4$. Unfortunately, it is not known if $\rho\tau\left(K_{2t}, \mathbf{Q}\left(t, \frac{n}{\omega(n)}\right)\right)$ exists.

Now we consider the general case. Using Theorem 1.6 and (9), we get the following corollary.

Corollary 4.11. *Suppose $p, q \geq 2$. Let $\omega(n) \to \infty$. If $\mathbf{Q}(p+1, n/q) \leq \mathbf{Q}\left(p, \frac{n}{\omega(n)}\right)$, then*

$$\rho\tau\left(K_{pq+1}, \mathbf{Q}\left(p, \frac{n}{\omega(n)}\right)\right) = \frac{q-1}{2q}.$$

Proof. The upper bound follows from Theorem 1.6 with $s = pq+1$ and $t = p$. The lower bound follows from (9) with $r = q$ and $t = p$:

$$\underline{\rho\tau}\left(K_{pq+1}, \mathbf{Q}\left(p+1, \frac{n}{q}\right)\right) \geq \frac{q-1}{2q}.$$ ∎

Now it is straightforward to see why Theorems 1.7, 1.8, 1.12, 1.13 and 1.14 are true. Theorem 1.6 and Construction 2.1 imply Theorem 1.7.

Proof of Theorem 1.7. Under the conditions of Theorem 1.7, using Theorem 1.6 and Construction 2.1, we have the following inequality:

$$\overline{\rho\tau}\left(K_s, \mathbf{Q}\left(t, \frac{n}{\omega(n)}\right)\right) \leq \frac{s-t-1}{2s-2} < \frac{r-1}{2r} \leq \underline{\rho\tau}(K_s, \mathbf{Q}(t,n)).$$

Trivially, $\frac{\mathbf{Q}(t,n)}{\omega(n)} \leq (1+o(1))\mathbf{Q}\left(t, \frac{n}{\omega(n)}\right)$. Therefore,

$$\overline{\rho\tau}\left(K_s, \frac{\mathbf{Q}(t,n)}{\omega(n)}\right) \leq \overline{\rho\tau}\left(K_s, \mathbf{Q}\left(t, \frac{n}{\omega(n)}\right)\right) < \underline{\rho\tau}(K_s, \mathbf{Q}(t,n)).$$ ∎

We use Theorem 4.6 to prove Theorem 1.8.

Proof of Theorem 1.8. Let $r = \left\lfloor \frac{s-1}{t} \right\rfloor$ and $f(n) = n2^{-\omega(n)\log^{\frac{r}{r+1}} n}$. To prove Theorem 1.8, we need that

$$\overline{\rho\tau}(K_s, \mathbf{Q}(t,f)) < \underline{\rho\tau}(K_s, \mathbf{Q}(t,n)). \tag{18}$$

We know that K_s has a phase transition from $\mathbf{Q}(t,n)$ to $\mathbf{Q}(t+1,n)$, i.e., $\overline{\rho\tau}(K_s, \mathbf{Q}(t+1,n)) < \underline{\rho\tau}(K_s, \mathbf{Q}(t,n))$. Therefore, to prove (18), it is sufficient to show that for $n \to \infty$, we have

$$\mathbf{RT}(n, K_s, \mathbf{Q}(t, f(n))) \leq \mathbf{RT}(n, K_s, \mathbf{Q}(t+1,n)) + o\big(n^2\big). \tag{19}$$

We may assume $\mathbf{Q}(t+1,n) \leq \mathbf{Q}(t, f(n))$ since otherwise we immediately have (19). Then, by $rt+1 \leq s \leq t(r+1)$, we can use Construction 2.1 with r and t as above and Theorem 4.6 with $p = t$ and $q = r+1$ to obtain that

$$\frac{r-1}{2r} \leq \underline{\rho\tau}(K_s, \mathbf{Q}(t+1,n)) \leq \overline{\rho\tau}(K_s, \mathbf{Q}(t,f)) \leq \frac{r-1}{2r}.$$

Hence $\overline{\rho\tau}(K_s, \mathbf{Q}(t,f)) = \underline{\rho\tau}(K_s, \mathbf{Q}(t+1,n))$, proving (19). ∎

Corollary 4.7 immediately yields Theorems 1.12 and 1.14.

Proof of Theorem 1.12. Let $p = t-1$ and $q = r+1$. Note that $p(q-1)+1 \leq s \leq pq$, so by Corollary 4.7 we get the desired result. ∎

Proof of Theorem 1.14. If Conjecture 1.10 is true for $\ell = t+1$, then for every $\varepsilon > 0$, we have

$$\mathbf{Q}(t+1,n) \leq \varphi_\varepsilon(n)\mathbf{Q}(t,n) \leq \mathbf{Q}(t, \varphi_\varepsilon(n)n), \tag{20}$$

where the second inequality holds by the definition of $\mathbf{Q}(t,n)$. Let $r = \lfloor (s-1)/t \rfloor$ and $\varepsilon = \frac{r}{r+1}$. Using the proof of Theorem 1.8 (or Corollary 4.7 with $p = t$ and $q = r+1$), we know that

$$\rho\tau(K_s, \mathbf{Q}(t+1,n)) = \rho\tau(K_s, \mathbf{Q}(t, \varphi_\varepsilon(n)n)). \tag{21}$$

Now combining (20) and (21), we have

$$\rho\tau(K_s, \mathbf{Q}(t+1,n)) = \rho\tau(K_s, \varphi_\varepsilon(n)\mathbf{Q}(t,n)),$$

which implies the desired result. ∎

Theorems 1.6 and 1.12 yield Theorem 1.13.

Proof of Theorem 1.13. Assume $r = \lfloor (s-1)/t \rfloor$, so $s < (r+1)t+1$, and therefore we have

$$\frac{s-t-1}{2s-2} < \frac{(r+1)t+1-t-1}{2((r+1)t+1-1)} = \frac{r}{2r+2}. \tag{22}$$

By Theorem 1.12 (here our t is $t-1$ in Theorem 1.12), we know that

$$\rho\tau(K_s, \mathbf{Q}(t+1,n)) = \frac{r-1}{2r}.$$

Then by Theorem 1.12 and the condition $\rho\tau(K_s, \mathbf{Q}(t+1,n)) < \rho\tau(K_s, \mathbf{Q}(t,n))$, we have for some $r' \geq r$ that

$$\rho\tau(K_s, \mathbf{Q}(t,n)) = \frac{r'}{2r'+2} \geq \frac{r}{2r+2}. \tag{23}$$

Now combining Theorem 1.6, (22) and (23), we have

$$\overline{\rho\tau}\left(K_s, \mathbf{Q}\left(t, \frac{n}{\omega(n)}\right)\right) \leq \frac{s-t-1}{2s-2} < \frac{r}{2r+2} \leq \rho\tau(K_s, \mathbf{Q}(t,n)).$$

By definition of $\mathbf{Q}(t,n)$, it is easy to see that $\frac{\mathbf{Q}(t,n)}{\omega(n)} \leq \mathbf{Q}\left(t, \frac{n}{\omega(n)}\right)$, thus

$$\overline{\rho\tau}\left(K_s, \frac{\mathbf{Q}(t,n)}{\omega(n)}\right) \leq \overline{\rho\tau}\left(K_s, \mathbf{Q}\left(t, \frac{n}{\omega(n)}\right)\right) < \rho\tau(K_s, \mathbf{Q}(t,n)).$$ ∎

5. Tools

The method of Dependent Random Choice was developed by Füredi, Gowers, Kostochka, Rödl, Sudakov, and possibly many others. The next lemma is taken from Alon, Krivelevich and Sudakov [19].

Lemma 5.1. *(Dependent Random Choice Lemma)*
Let a, d, m, n, r be positive integers. Let $G = (V, E)$ be a graph with n vertices and average degree $d = 2e(G)/n$. If there is a positive integer t such that

$$\frac{d^t}{n^{t-1}} - \binom{n}{r}\left(\frac{m}{n}\right)^t \geq a, \tag{24}$$

then G contains a subset U of at least a vertices such that every r vertices in U have at least m common neighbors.

Conlon, Fox, and Sudakov [20] extended Lemma 5.1 to hypergraphs. The *weight* $w(S)$ of a set S of edges in a hypergraph is the number of vertices in the union of these edges.

Lemma 5.2. *(Hypergraph Dependent Random Choice Lemma).*

Suppose s, Δ are positive integers, $\varepsilon, \delta > 0$, and $G_r = (V_1, \dots, V_r; E)$ is an r-uniform r-partite hypergraph with $|V_1| = \dots = |V_r| = N$ and at least εN^r edges. Then there exists an $(r-1)$-uniform $(r-1)$-partite hypergraph G_{r-1} on the vertex sets $V_2, \dots, V_r$ which has at least $\frac{\varepsilon^s}{2} N^{r-1}$ edges and such that for each nonnegative integer $w \le (r-1)\Delta$, there are at most $4r\Delta\varepsilon^{-s}\beta^s w^{r\Delta} r^w N^w$ dangerous sets of edges of G_{r-1} with weight w, where a set S of edges of G_{r-1} is dangerous if $|S| \le \Delta$ and the number of vertices $v \in V_1$ such that for every edge $e \in S, e + v \in G_r$ is less than βN.

6. Proofs of Theorems 4.1 and 4.2

Here we first provide two proofs of Theorem 4.1 using Lemma 5.1. The structures of these two proofs are similar, but we use Lemma 5.1 in somewhat different ways. We suppose that there exist K_5-free graphs G_n with $e(G_n) > \varepsilon n^2$ for any n sufficiently large. Next we use Lemma 5.1 to find a K_5 in G_n. Both proofs show that $e(G_n) = o(n^2)$, but Proof I gives a better bound.

From (12) we know that every triangle-free graph G_n contains an independent set of size at least $\big(1/\sqrt{2} - o(1)\big)\sqrt{n \log n}$. We will use this in both proofs.

Proof I of Theorem 4.1. Let $\varepsilon_n = \omega(n)^{-1/4}$. Assume that there is a K_5-free graph G_n with

$$e(G_n) \ge \varepsilon_n n^2 \qquad \text{and} \qquad \alpha(G_n) < \frac{\sqrt{n \log n}}{\omega(n)}. \tag{25}$$

We apply Lemma 5.1 to G_n with

$$a = \frac{4n}{\omega(n)^2}, \qquad r = 3, \qquad d = 2\varepsilon_n n, \qquad m = \sqrt{n \log n} \qquad \text{and} \qquad t = 7.$$

Now the condition of Lemma 5.1, (24) is satisfied as

$$\frac{d^t}{n^{t-1}} - \binom{n}{r}\left(\frac{m}{n}\right)^t \ge (2\varepsilon_n)^7 n - n^3\left(\frac{\log n}{n}\right)^{7/2} > \varepsilon_n^7 n \ge \frac{n}{\omega(n)^{7/4}} > a.$$

So there exists a vertex subset U of G with $|U| = a = 4n/\omega(n)^2$ such that all subsets of U of size 3 have at least m common neighbors. Either U has an independent set of size at least $\left(\frac{1}{\sqrt{2}} - o(1)\right)\sqrt{\frac{4n}{\omega(n)^2}\log\left(\frac{4n}{\omega(n)^2}\right)} > \alpha(G_n)$, or $G_n[U]$ contains a triangle. In the latter case, denote by W the common neighborhood of the vertices of this triangle. It follows that $|W| \ge m = \sqrt{n \log n} > \alpha(G_n)$, so $G_n[W]$ contains an edge, and this edge forms a K_5 with the triangle. ∎

Proof II of Theorem 4.1. Let $\varepsilon_n = \frac{\log\log(\omega(n)/2)}{\log(\omega(n)/2)}$. Assume (25) again. We shall apply Lemma 5.1 to G_n with

$$a = \frac{\sqrt{n \log n}}{\omega(n)}, \qquad r = 2, \qquad d = 2\varepsilon_n n, \qquad m = \frac{4n}{\omega(n)^2} \qquad \text{and} \qquad t = \frac{\log n}{\log(\omega(n)/2)}.$$

Now when n, and hence $\omega(n)$ is sufficiently large, we have

$$\frac{d^t}{n^{t-1}} = (2\varepsilon_n)^t n = n \cdot 2^{-\log(\frac{1}{2\varepsilon_n})\cdot\frac{\log n}{\log(\omega(n)/2)}} > n^{1-\varepsilon_n} \text{ and } \left(\frac{\omega(n)}{2}\right)^{2t} = 2^{2\log\left(\frac{\omega(n)}{2}\right)\cdot\frac{\log n}{\log(\omega(n)/2)}} = n^2,$$

yielding that (24) is satisfied:

$$\frac{d^t}{n^{t-1}} - \binom{n}{r}\left(\frac{m}{n}\right)^t \geq n^{1-\varepsilon_n} - \frac{n^2}{2n^2} > a.$$

This means that there is a vertex subset U of G_n with $|U| = a$ such that every pair of vertices in U has at least m common neighbors. We chose $a > \alpha(G_n)$, so $G_n[U]$ contains an edge u_1u_2. Let $W := N(u_1) \cap N(u_2)$. Since G_n is K_5-free, $G_n[W]$ does not contain a triangle. Then W contains an independent set of size at least $\big(1/\sqrt{2} - o(1)\big)\sqrt{|W|\log|W|} \geq \big(1/\sqrt{2} - o(1)\big)\sqrt{m\log m} > \alpha(G_n)$. ∎

The proof of Theorem 4.2 is very similar to the first proof of Theorem 4.1.

Proof of Theorem 4.2. Let

$$\begin{aligned} &r = p, \qquad t = 2p/\vartheta, \qquad \varepsilon_n = \omega(n)^{-1/t}, \qquad a = n/\omega(n),\\ &d = 2\varepsilon_n n \qquad \text{and} \qquad m = \mathbf{R}(p-1, \mathbf{Q}(p, n/\omega(n))) < n^{1-\vartheta}. \end{aligned}$$

Assume that there is a K_{2p-1}-free graph G_n with $e(G_n) \geq \varepsilon_n n^2$ and $\alpha(G_n) < \mathbf{Q}\Big(p, \frac{n}{\omega(n)}\Big)$. We check (24), i.e., that

$$\frac{d^t}{n^{t-1}} - \binom{n}{r}\frac{m^t}{n^t} \geq (2\varepsilon_n)^t n - n^r \cdot n^{-\vartheta t} > \varepsilon_n^t n = a.$$

Therefore, we can apply Lemma 5.1 to G_n with the parameters a, d, m, n, r, t as above to find a $U \subset V(G_n)$ with $|U| = a$ such that all subsets of U of size r have at least m common neighbors. The set U does not contain an independent set of size $\mathbf{Q}(p, n/\omega(n))$, so $G_n[U]$ contains a K_p. Denote by W the common neighborhood of the vertices of this K_p. It follows that $|W| \geq m$. Then since $G_n[W]$ does not contain an independent set of size $\mathbf{Q}(p, n/\omega(n))$, it contains a K_{p-1}, which together with K_p forms a K_{2p-1}. ∎

7. Proofs of Theorems 4.6 and 4.9

The proofs of Theorems 4.3 and 4.6 are very similar, therefore the proof of Theorem 4.3 is put into the Appendix.

We start by sketching the proof of Theorem 4.6. Suppose that G_n has more than $\Big(\frac{q-2}{q-1} + \delta\Big)\frac{n^2}{2}$ edges and is K_{pq}-free, then we apply Szemerédi's Regularity Lemma to G_n and find a K_q in the cluster graph R_k (see below). Let $V_1, \dots, V_q$ be the vertices of a q-clique in the cluster graph. We use Lemma 5.1 to find a K_{2p} in $V_{q-1} \cup V_q$ and use Lemma 5.2 to find a K_p in each V_i for $1 \le i \le q-2$ such that these cliques together form a K_{pq} in G_n. The details are below.

PROOF OF THEOREM 4.6. Suppose to the contrary that there is a K_{pq}-free graph G_n with n sufficiently large,

$$e(G_n) \ge \left(\frac{q-2}{q-1} + \delta\right)\frac{n^2}{2} \qquad \text{and} \qquad \alpha(G_n) < \mathbf{Q}(p, f(n)).$$

We apply Szemerédi's Regularity Lemma to G_n with regularity parameter $\rho = \delta/2^{2^q}$ to get a cluster graph R_k on k vertices where the vertices of R_k are the clusters of the Szemerédi Partition, and adjacent if the pair is ρ-regular and has density at least $\delta/2$. It is standard to check that the number of edges of R_k is at least $\left(\frac{q-2}{q-1} + \frac{\delta}{2}\right)\frac{k^2}{2}$. So, by Turán's Theorem R_k contains a K_q, and by Claim 7.1, we can find a K_{pq} in G_n, a contradiction.

To complete the proof, it is sufficient to prove the following assertion.

Claim 7.1. *If $\alpha(G_n) < \mathbf{Q}\left(p, n2^{-\omega(n)\log^{1-1/q} n}\right)$ and there exists a K_q in a cluster graph of G_n, then we can find a K_{pq} in G_n.*

There exist q vertices in R_k, denoted by $V_1, \dots, V_q$, that induce a K_q. We define a q-uniform q-partite hypergraph H^0 whose vertex set is $\bigcup V_i$ and edge set $E(H^0)$ is the family of q-sets that span q-cliques in G_n and contain one vertex from each of $V_1, \dots, V_q$. Let $N = |V_i| = n/k$, then by the counting lemma, $|E(H^0)| \ge \varepsilon_0 N^q$, where $\varepsilon_0 > (\delta/3)^{\binom{q}{2}}$. Let

$$\beta = f(n)/N, \quad s = \log^{\frac{1}{q}} n, \quad \varepsilon_i = \varepsilon_0^{\log^{\frac{i}{q}} n} 2^{-\frac{s^i - 1}{s-1}}, \quad r_i = q - i, \quad \Delta_i = p^{r_i} \quad \text{and} \quad w_i = pr_i.$$

We start from H^0. For $1 \le i \le q-2$ we apply Lemma 5.2 to H^{i-1} with $\Delta = \Delta_i, \varepsilon = \varepsilon_{i-1}, r = r_{i-1}$ and $w = w_i$ to get H^i. Note that $\Delta, \varepsilon_0, r, w$ and k are all constants. It is easy to check that for $1 \le i \le q-2$, we have

$$\begin{aligned} 4r\Delta\varepsilon^{-s}\beta^s w^{r\Delta} r^w N^w &= O\left(2^{2\log^{\frac{i-1}{q}} n} \varepsilon_0^{-\log^{\frac{i}{q}} n} k^{\log^{\frac{1}{q}} n} 2^{-\omega(n)\log n} N^w\right) \\ &= O(n^{-\omega(n)/2}) = o(1) < 1. \end{aligned}$$

Then by Lemma 5.2 there exists an r_i-uniform r_i-partite hypergraph H^i on the vertex sets $V_{i+1}, \dots, V_q$ that contains at least $\varepsilon_i N^{r_i}$ edges and contains no dangerous sets of Δ_i edges on w_i vertices (Recall that a set S of Δ_i edges on w_i vertices is dangerous if the number of vertices $v \in V_i$ such that for every edge $e \in S, e + v \in H^{i-1}$ is less than βN). Now we have a hypergraph sequence $\{H^\ell\}_{\ell=0}^{q-2}$. We will prove by induction on i that there is a p-set $A^{q-\ell} \subset V_{q-\ell}$ for $0 \le \ell \le i$ such that $G_n\left[A^{q-\ell}\right] = K_p$ and $H^{q-i-1}\left[\bigcup_{\ell=0}^{i} A^{q-\ell}\right]$ is complete r_{q-i-1}-partite. Note that if a vertex set T is an edge of H^0, then $G_n[T]$ is a q-clique. So $G_n\left[\bigcup_{\ell=0}^{q-1} A^{q-\ell}\right] = K_{pq}$, which will prove Claim 7.1.

We first show that the induction hypothesis holds for $i = 1$. Note that $r_{q-2} = 2$, so H^{q-2} is a bipartite graph on $2N$ vertices with at least $\varepsilon_{q-2}N^2$ edges. We now apply Lemma 5.1 to H^{q-2} with

$$a = 2\beta N, \qquad d = \varepsilon_{q-2}N, \qquad t = s, \qquad r = p \qquad \text{and} \qquad m = \beta N.$$

We check condition (24):

$$\begin{aligned}\frac{(\varepsilon_{q-2}N)^s}{(2N)^{s-1}} - \binom{2N}{p}\left(\frac{\beta N}{2N}\right)^s &\geq (\varepsilon_0/2)^{\log^{1-1/q} n}N - n^p k^s 2^{-\omega(n)\log n} \\ &= (\varepsilon_0/2)^{\log^{1-1/q} n}N - o(1) \geq 2\beta N.\end{aligned}$$

Therefore we have a subset U of $V_{q-1} \cup V_q$ with $|U| = 2\beta N$ such that every p vertices in U have at least βN common neighbors in H^{q-2}. Either V_{q-1} or V_q contains at least half of the vertices of U, so w.l.o.g. we may assume that $U' = U \cap V_{q-1}$ contains at least $\beta N = m$ vertices. Because $\alpha(G_n) < \mathbf{Q}(p, m)$, the vertex set U' contains a p-vertex set A^{q-1} such that $G_n[A^{q-1}] = K_p$. The vertices of A^{q-1} have at least m common neighbors in V_q, so their common neighborhood also contains a p-vertex subset A^q of V_q such that $G_n[A^q] = K_p$. Now $H^{q-2}[A^{q-1} \cup A^q]$ is complete bipartite. We are done with the base case $i = 1$.

For the induction step, assume that the induction hypothesis holds for $i - 1$, then we can find a complete r_{q-i}-partite subhypergraph $\widetilde{H}^{q-i}$ of H^{q-i} spanned by $\bigcup_{\ell=0}^{i-1} A^{q-\ell}$, where $G_n[A^{q-\ell}] = K_p$ for every ℓ. The hypergraph H^{q-i} has no dangerous set of Δ_{q-i} edges on w_{q-i} vertices, and $\widetilde{H}^{q-i}$ contains $pi = w_{q-i}$ vertices and $p^i = \Delta_{q-i}$ edges, so $\widetilde{H}^{q-i}$ is not dangerous. Then we can find a set B of βN vertices in V_{q-i} such that for every edge $e \in \widetilde{H}^{q-i}$ and every vertex $v \in B$, $e + v \in H^{q-i-1}$, which means $H^{q-i-1}\Big[B \cup \bigcup_{\ell=0}^{i-1} A^{q-\ell}\Big]$ is complete r_{q-i-1}-partite. Then, because $\alpha(G_n) < \mathbf{Q}(p, \beta N)$, we can find a p-vertex subset A^{q-i} of B such that $G_n[A^{q-i}] = K_p$. ∎

The proof of Theorem 4.9 is a combination of Claim 7.1 and an easy application of Szemerédi's Regularity Lemma, (see the Appendix of Balogh-Lenz [8] for similar proofs). The idea is that instead of proving only that the cluster graph is K_q-free, like in the proof of Theorem 4.6, we also bound the density of regular pairs.

Proof of Theorem 4.9. Given $\varepsilon > 0$, let $\rho = \varepsilon/2^{2^t}$ and $M = M(\rho) > 1/\rho$ be the upper bound on the number of partitions guaranteed by Szemerédi's Regularity Lemma with regularity parameter ρ. Suppose we have a K_{2t}-free graph G_n with

$$e(G_n) \geq \left(\frac{(t-1)(q-2)}{t(q-1)} + \varepsilon\right)\frac{n^2}{2} \qquad \text{and} \qquad \alpha(G_n) < \mathbf{Q}\Big(t, \frac{\varepsilon n}{M}\Big).$$

We apply Szemerédi's Regularity Lemma to G_n with regularity parameter ρ to get a cluster graph R_k on $k \leq M$ vertices where two vertices are adjacent if the pair is ρ-regular and has density at least $\varepsilon/2$. It is standard to check that more than $\left(\frac{(t-1)(q-2)}{t(q-1)} + \frac{\varepsilon}{2}\right)\frac{n^2}{2}$ edges of G_n are between pairs of classes that are ρ-regular and have density at least $\varepsilon/2$.

Assume that the density d of a ρ-regular pair (V_i, V_j) is at least $\frac{t-1}{t}+\varepsilon$. Because $\alpha(G_n) < \mathbf{Q}(t, \varepsilon n/M)$ and $|V_i| \geq \varepsilon n/M$, there is a t-clique in V_i, each of whose vertices has at least $(d-\rho)|V_j| \geq (\frac{t-1}{t}+\frac{\varepsilon}{2})|V_j|$ neighbors in V_j, hence vertices of this t-clique have at least $\varepsilon|V_j|$ common neighbors. Then we can find a t-clique in their common neighborhood since $\alpha(G_n) < \mathbf{Q}(t, \varepsilon n/M)$ and $\varepsilon|V_j| \geq \varepsilon n/M$. Thus we find a K_{2t} in G_n, a contradiction. Therefore the density of any ρ-regular pair is at most $\frac{t-1}{t}+\varepsilon$. Then R_k has at least

$$\left(\frac{(t-1)(q-2)}{t(q-1)}+\frac{\varepsilon}{2}\right)\frac{n^2}{2}\cdot\left(\left(\frac{t-1}{t}+\varepsilon\right)\left(\frac{n}{k}\right)^2\right)^{-1} > \left(\frac{q-2}{q-1}+\frac{\varepsilon}{4}\right)\frac{k^2}{2}$$

edges, so there is a K_q in R_k. Then, by Claim 7.1, there is a K_{pq} in G_n, which is a contradiction. ∎

8. Concluding remarks and open problems

First, using the Dependent Random Choice Lemma, we prove that K_ℓ-free graphs with small independence number are sparse.

Theorem 8.1. *Let $\ell \geq 3$ be an integer and $s = \lceil \ell/2 \rceil$. Fix a positive constant $c < \frac{1}{s(s-1)}$. Let $G_{n,\ell}$ be a graph on n vertices not containing K_ℓ.*

$$\textit{If} \qquad \alpha(G_{n,\ell}) < \mathbf{Q}(\ell, n)n^c, \qquad \textit{then} \qquad e(G_{n,\ell}) = o(n^2).$$

Proof. The general bound (2) on Ramsey numbers implies that there exists a constant $\vartheta > 0$ (depending on ℓ and c) such that $\mathbf{R}(s, \mathbf{Q}(\ell, n)n^c) < n^{1-\vartheta}$. Assume that $G = G_{n,\ell}$ has more than εn^2 edges and $\varepsilon > n^{-\vartheta^2/2s}$. We apply Lemma 5.1 to G with

$$r = s, \qquad d = 2\varepsilon n, \qquad t = 2s/\vartheta \qquad \text{and} \qquad a = m = \mathbf{R}(s, \mathbf{Q}(\ell, n)n^c).$$

Now the condition of Lemma 5.1, (24) is satisfied as

$$\frac{d^t}{n^{t-1}} - \binom{n}{r}\frac{m^t}{n^t} > (2\varepsilon)^t n - n^s \cdot n^{-\vartheta\cdot 2s/\vartheta} > \varepsilon^t n > n^{1-\frac{\vartheta^2}{2s}\cdot\frac{2s}{\vartheta}} > a.$$

Therefore we can use Lemma 5.1 (with the parameters a, d, m, r, t as above) to find a vertex subset U of G with $|U| = a$ such that all subsets of U of size r have at least m common neighbors. The set U does not contain an independent set of size $\mathbf{Q}(\ell, n)n^c$, so $H_{n,\ell}[U]$ contains a K_s. Denote by W the common neighborhood of the vertices of this K_s. It follows that $|W| \geq m$. Then $H_{n,\ell}[W]$ also contains a K_s, which together with the K_s found in $H_{n,\ell}[U]$ forms a K_{2s}. ∎

Next we propose some problems. We proved that $\rho\tau\big(K_5, o\big(\sqrt{n\log n}\big)\big) = 0$, and it was known that $\rho\tau(K_5, \mathbf{Q}(3, n/2)) = 1/4$. It would be interesting to know if there is any sharper transition of K_5 at $c \cdot \mathbf{Q}(3, n/2)$ for $c < 1$, hence it is natural to propose the following two problems:

Question 8.2. Determine $\mathbf{RT}(n, K_5, (1-\varepsilon)\mathbf{Q}(3, n/2))$.

Question 8.3. Determine $\mathbf{RT}(n, K_5, c \cdot \mathbf{Q}(3, n/2))$ for $0 < c < 1$.

We proved that if Conjecture 1.10 is true, then $\overline{\rho\tau}(K_{2t}, o(\mathbf{Q}(t,n))) \le \frac{t-1}{4t}$. Note that the Bollobás-Erdős graph gave matching lower bound for $t = 2$, so finding constructions to give matching lower bounds on $\underline{\rho\tau}(K_{2t}, o(\mathbf{Q}(t,n)))$ could be a very challenging problem. Probably a construction, if exists, is an extension of the Bollobás-Erdős graph. There are generalizations of the Bollobás-Erdős graph in [6–8, 21]. The simplest open case is stated below:

Question 8.4. Determine $\overline{\rho\tau}\big(K_6, o\big(\sqrt{n\log n}\big)\big)$ and $\underline{\rho\tau}\big(K_6, o\big(\sqrt{n\log n}\big)\big)$.

We have 1/6 as an upper bound. Sudakov proved that $\rho\tau(K_6, f) = 0$ for $f(n) = \mathbf{Q}\Big(3, n2^{-\omega(n)\sqrt{\log n}}\Big)$, but it is not clear what happens when $f(n)$ is between $\mathbf{Q}\Big(3, n2^{-\omega(n)\sqrt{\log n}}\Big)$ and $o\big(\sqrt{n\log n}\big)$. In particular, we would like to know the answer to the following question:

Question 8.5. At which function $f(n)$ does K_6 have a strong phase transition to 0, i.e., $0 = \overline{\rho\tau}(K_6, o(f)) < \underline{\rho\tau}(K_6, f)$?

Another surprising phenomenon is that $\rho\tau(K_4, o(\sqrt{n\log n})) = 0 = \rho\tau(K_5, o(\sqrt{n\log n}))$. We know that $\overline{\rho\tau}(K_6, o(\sqrt{n\log n})) \le 1/6 < 1/4 = \rho\tau(K_7, o(\sqrt{n\log n}))$. It would be interesting to know if

$$\rho\tau\Big(K_7, o\Big(\sqrt{n\log n}\Big)\Big) = \overline{\rho\tau}\Big(K_8, o\Big(\sqrt{n\log n}\Big)\Big).$$

Appendix A. Proof of Theorem 4.3

This proof is very similar to that of Theorem 4.6 in Section 7, so we skip some details. Suppose to the contrary that there is a K_{pq-1}-free graph G_n with n sufficiently large,

$$e(G_n) \ge \left(\frac{q-2}{q-1} + \delta\right)\frac{n^2}{2} \qquad \text{and} \qquad \alpha(G_n) < \mathbf{Q}(p, f(n)).$$

Just as what we did in the proof of Theorem 4.6, we apply Szemerédi's Regularity Lemma to G_n with regularity parameter $\rho = \delta/2^{2^q}$ to get a cluster graph R on k vertices. Similarly to the proof of Theorem 4.6, we can find q vertices $V_1, \ldots, V_q$ that span a K_q in R. Now consider a q-uniform q-partite hypergraph H^0 whose vertex set is $\bigcup V_i$ and edge set $E(H^0)$ is the family of q-sets that span q-cliques in G_n and contain one vertex from each of $V_1, \ldots, V_q$. Let $N = |V_i| = n/k$, then by the counting lemma, $|E(H^0)| \ge \varepsilon_0 N^q$, where $\varepsilon_0 > (\delta/3)^{\binom{q}{2}}$. Let

$$\beta = f(n)/N,\ s = \log^{\frac{1}{q-1}} n,\ \varepsilon_i = \varepsilon_0^{s^i}/2^{\frac{s^i-1}{s-1}},\ r_i = q-i,\ w_i = pr_i - 1 \text{ and } \Delta_i = p^{r_i-1}(p-1).$$

We start from H^0. For $1 \le i \le q-2$ we apply Lemma 5.2 to H^{i-1} with $\Delta = \Delta_i, \varepsilon = \varepsilon_{i-1}, r = r_{i-1}$ and $w = w_i$ to get H^i. It is easy to check that for $1 \le i \le q-2$, we have

$$\begin{aligned} 4r\Delta\varepsilon^{-s}\beta^s w^{r\Delta} r^w N^w &= O\left(2^{2\log^{\frac{i-1}{q-1}} n}\varepsilon_0^{-\log^{\frac{i}{q-1}} n} k^s 2^{-\omega(n)\log n} N^w\right) \\ &= O(n^{-\omega(n)/2}) = o(1) < 1. \end{aligned}$$

Then by Lemma 5.2 there exists an r_i-uniform r_i-partite hypergraph H^i on the vertex sets $V_{i+1}, \dots, V_q$ that contains at least $\varepsilon_i N^{r_i}$ edges and contains no dangerous set of Δ_i edges on w_i vertices.

Note that $r_{q-2} = 2$, so H^{q-2} is a bipartite graph on $2N$ vertices with at least $\varepsilon_{q-2}N^2$ edges. We now apply Lemma 5.1 to H^{q-2} with

$$a = 2\beta N, \qquad d = \varepsilon_{q-2}N, \qquad t = 2p/\vartheta, \qquad r = p \qquad \text{and} \qquad m = \mathbf{R}(p-1, \mathbf{Q}(p, f(n))).$$

Note that $m < n^{1-\vartheta}$. We check condition (24):

$$\begin{aligned} \frac{(\varepsilon_{q-2}N)^t}{(2N)^{t-1}} - \binom{2N}{p}\left(\frac{m}{2N}\right)^t &\ge (\varepsilon_0/2)^{t\log^{\frac{q-2}{q-1}} n} N - n^p k^t n^{-2p} \\ &= (\varepsilon_0/2)^{t\log^{\frac{q-2}{q-1}} n} N - o(1) \ge a. \end{aligned}$$

Therefore we have a subset U of $V_{q-1} \cup V_q$ with $|U| = 2\beta N$ such that every p vertices in U have at least m common neighbors in H^{q-2}. Either V_{q-1} or V_q contains at least half of the vertices of U, so w.l.o.g. we may assume that $U' := U \cap V_{q-1}$ contains at least βN vertices. Because $\alpha(G_n) < \mathbf{Q}(p, \beta N)$, the vertex set U' contains a p-vertex set A^{q-1} such that $G_n[A^{q-1}] = K_p$. The vertices of A^{q-1} have at least $m = \mathbf{R}(p-1, \mathbf{Q}(p, \beta N))$ common neighbors in V_q, so their common neighborhood contains a $(p-1)$-vertex subset A^q of V_q such that $G_n[A^q] = K_{p-1}$. Now $H^{q-2}[A^{q-1} \cup A^q]$ is complete bipartite. Then similarly to the proof of Theorem 4.6, for $1 \le i \le q$, we can find a subset A^i of V_i satisfying the following conditions:

- $G_n[A^q] = K_{p-1}$.
- For $1 \le i < q$, $G_n[A^i] = K_p$.
- $H^0[\bigcup_{i=1}^q A^i]$ is complete q-partite.

If a vertex set T is an edge of H^0, then $G_n[T] = K_q$. So $G_n[\bigcup_{i=1}^q A^i] = K_{pq-1}$, which is a contradiction.

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