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LONGEST CYCLES IN 3-CONNECTED 3-REGULAR GRAPHS

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Introduction. In this paper, we study the following question: How long a cycle must there be in a 3-connected 3-regular graph on n vertices? For planar graphs this question goes back to Tait [**6**], who conjectured that any planar 3-connected 3-regular graph is hamiltonian. Tutte [**7**] disproved this conjecture by finding a counterexample on 46 vertices. Using Tutte's example, Grünbaum and Motzkin [**3**] constructed an infinite family of 3-connected 3-regular planar graphs such that the length of a longest cycle in each member of the family is at most n^c, where $c = 1 - 2^{-17}$ and n is the number of vertices. The exponent c was subsequently reduced by Walther [**8, 9**] and by Grünbaum and Walther [**4**].

It is natural to ask what one can say when the planarity condition is dropped. For 2-connected 3-regular graphs, Bondy and Entringer [**2**] proved that the length of a longest cycle is at least $4 \log_2 n - 4 \log_2 \log_2 n - 20$, and an example due to Lang and Walther [**5**] shows that this result is essentially best possible.

Let $f(n)$ denote the largest integer k such that every 3-connected 3-regular graph on n vertices contains a cycle of length at least k. For planar graphs, Barnette [**1**] proved that

$$f(n) \geqq 3 \log_2 n - 10,$$

a result which, as noted above, has been improved under the weaker condition of 2-connectedness.

Here, we shall prove that

$$(1) \quad e^{c_1\sqrt{\log_e n}} \leqq f(n) \leqq c_2 n^{\log 8/\log 9},$$

where c_1 and c_2 are appropriate constants.

The upper bound in (1) is obtained by means of a construction similar to those described in [**3, 4, 8, 9**] but we use the Petersen graph instead of other, planar, graphs.

Construction. Let P_0 denote the Petersen graph. We construct a sequence of graphs $P_1, P_2, \ldots, P_k, \ldots$ recursively, as follows:

Assume that we have already constructed P_k. If P_k has n vertices, let us enumerate them as $v_1, v_2, \ldots, v_n$. We replace each edge of P by a path of length three so that, in the resulting graph, v_i has three neighbours,

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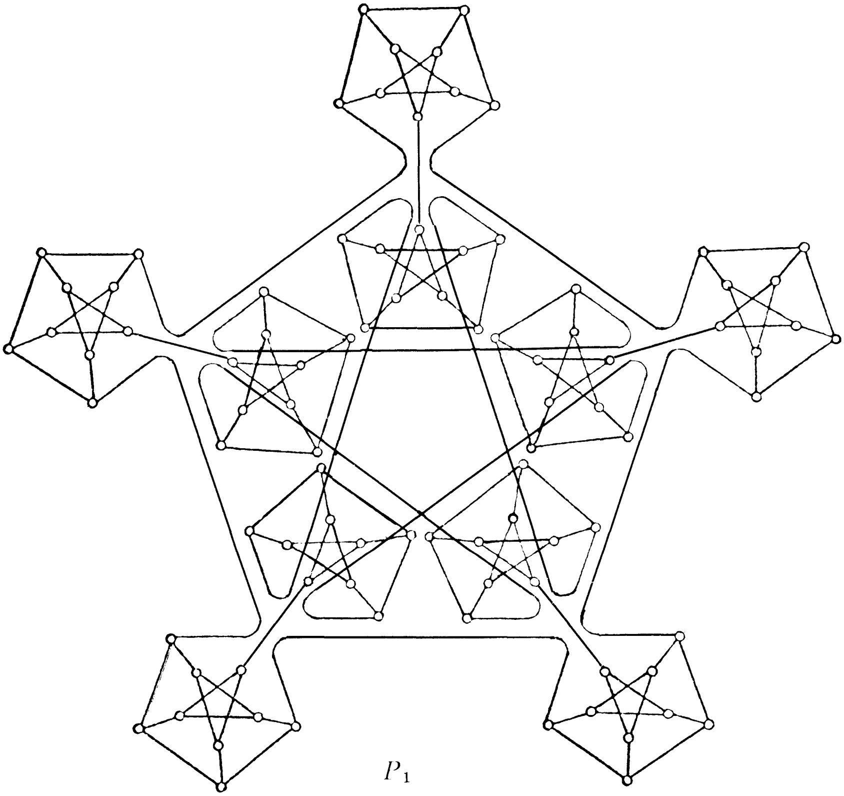


P_1

FIGURE 1

a_i, b_i and c_i. We now omit the original vertices $v_1, v_2, \ldots, v_n$, take n copies of the Petersen graph, delete one vertex from each of them, and identify the three vertices of degree two in the ith copy with the vertices a_i, b_i and c_i (in any order). The resulting graph is P_{k+1}. (P_1 is depicted in figure 1.)

It follows easily, by induction on k, that P_k is both 3-connected and 3-regular. Now the number of vertices in P_{k+1} is $9n$. On the other hand, if the length of a longest cycle in P_k is l, then P_{k+1} has no cycle of length greater than $8l$ since a cycle in P_{k+1} can visit at most l of the truncated Petersen graphs and can visit at most 8 vertices of each (because the Petersen graph is nonhamiltonian). Therefore $n = 10.9^k$ and $l \leqq 9.8^k$. This establishes the upper bound in (1) for integers of the form 10.9^k. Any even integer n not of this form may be expressed as

$$n = 10.9^k + 8s + 2t$$

where

$$0 \leqq s < 10.9^k \text{ and } 0 \leqq t \leqq 3.$$

An appropriate graph on n vertices can then be constructed from P_k by replacing s of its vertices by truncated Petersen graphs (as above) and then inflating t vertices into triangles.

The following theorem establishes the lower bound in (1).

THEOREM. *If G is a 3-connected 3-regular graph on n vertices, then it contains a cycle of length at least*

$$g(n) = e^{c\sqrt{\log_e n}}$$

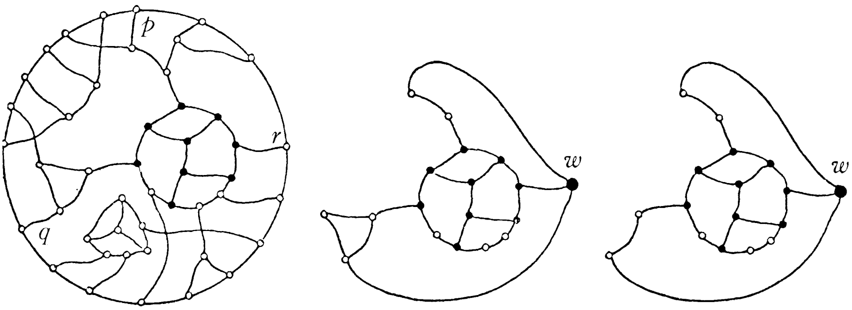
where $c^2 = \frac{2}{3}\log_e \frac{3}{2}$.

In the proof we shall use the fact that, if $n \geqq 4$, then

$$(2) \quad g\left(\frac{6n}{g^3(n)}\right) \geqq \frac{2}{3}g(n).$$

Proof. We shall use induction on n. For $n = 4$, $G = K_4$ and $g(4) < 4$, so the theorem is trivial. Assume that it holds for all graphs on at most $n - 1$ vertices, where $n > 4$, and let G be a 3-connected 3-regular graph on n vertices.

Let C be a longest cycle in G, of length l, and let S denote the set of vertices of G not on C. Since G is 3-connected, each vertex x of S is connected to C by three paths $P(x)$, $Q(x)$ and $R(x)$, having only the vertex x in common. Let $p(x)$, $q(x)$ and $r(x)$ denote the respective terminal vertices of these paths. We now define an equivalence relation on S by calling x and y equivalent if the sets $\{p(x), q(x), r(x)\}$ and $\{p(y), q(y), r(y)\}$ are the same. Since $p(x)$, $q(x)$ and $r(x)$ are not, in general, uniquely determined by x, a number of such equivalence relations may be so defined. For each of these equivalence relations, we consider all of the associated equivalence classes. We denote by W the largest such equivalence class, and by p, q, and r the corresponding terminal vertices. The situation is illustrated in figure 2(a), with the vertices of W indicated in black.



G (a) H (b) H′ (c)

FIGURE 2

Clearly

$$(3)\quad |W| \geqq (n - l)/\binom{l}{3}.$$

Consider the subgraph of G formed by taking the union of the paths $P(x)$, $Q(x)$ and $R(x)$, where x runs through W. In this graph, p, q and r have degree one. We identify them to form a new vertex w of degree three, and call the resulting graph H. (See figure 2(b).) Let $W^* = W \cup \{w\}$. We claim that any two vertices of W^* are connected in H by three internally-disjoint paths. For suppose that u and v are two vertices of W^* that are not connected by three internally-disjoint paths. We distinguish two cases, depending on whether or not u and v are adjacent.

If u and v are nonadjacent, then they are separated by a 2-vertex cut $\{x, y\}$. Clearly $w \notin \{u, v\}$. In fact, $w \in \{x, y\}$, for otherwise w would be separated by $\{x, y\}$ from at least one of u and v and, since both u and v are connected to w by three internally-disjoint paths, this is impossible. Without loss of generality, suppose that $x = w$, and consider the subgraph $H - y$ (in which w is a cut vertex separating u and v). There are two internally-disjoint paths from u to w in the block of $H - y$ containing u and two from v to w in the block containing v. But this implies that the degree of w is at least four, a contradiction.

A similar contradiction is reached in the case when u and v are adjacent. In fact, if we denote the edge joining u and v by y and the cut vertex of $H - y$ by x, then the above argument remains valid, word for word.

Therefore any two vertices of W^* are indeed connected by three internally-disjoint paths. We now consider a connected subgraph H' of H which contains all the vertices of W^* and in which any two vertices of W^* are connected by three internally-disjoint paths. We choose H' so that it has as few edges as possible subject to these conditions. (See figure 2(c).) We claim that all the vertices of H′ not belonging to W^* have degree two in H'. Let z be such a vertex. By the maximality of W, z is not connected to w by three internally-disjoint paths in H'. If z and w are nonadjacent, we can find a 2-vertex cut $\{x, y\}$ separating z and w. Let Z denote the set of all vertices of H' separated by $\{x, y\}$ from w. Clearly, $W^* \cap Z$ is empty. Since $Z \subset V(H')$ and H' has as few edges as possible, there must be an (x, y)-path P' in H' all of whose internal vertices belong to Z. (See figure 3.)

Let H'' be the subgraph $(H' - Z) \cup P'$ of H'. Then H'' clearly has all of the properties required of H', and so, by the choice of H', we must have $H'' = H'$. But this implies that z has degree two in H'. A similar argument applies in the case when z and w are adjacent. It follows that each vertex of H' not in W^* has degree two in H'.

Thus H' is a subdivision of a 3-connected 3-regular graph H^* with

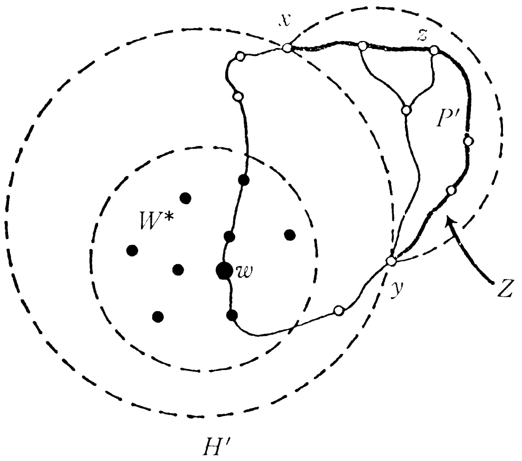


FIGURE 3

vertex set W^*. By (3),

$$|W^*| = |W| + 1 \geqq \frac{n - l}{\binom{l}{3}} + 1 > \frac{6n}{l^3}.$$

If $l \geqq g(n)$, then the theorem is proved. If not, then, by the induction hypothesis, H^* contains a cycle C^* of length l^*, where

$$l^* \geqq g\left(\frac{6n}{l^3}\right) \geqq g\left(\frac{6n}{g^3(n)}\right).$$

Using (2), we obtain

(4) $l^* \geqq \frac{2}{3}g(n) > \frac{2}{3}l.$

There are two cases: either C^* contains the vertex w or it does not.

Suppose first that $w \in V(C^*)$. Then two of p, q and r are connected in G by a path P^* of length at least l^*. By the maximality of C, $l \geqq 2l^*$, which contradicts (4).

Next, suppose that $w \notin V(C^*)$. Then, corresponding to C^*, there is a cycle C' in G, disjoint from C. Since G is 3-connected, there exist three disjoint paths connecting C and C'. Now we can choose two of them, joining an $x \in C$ to an $x' \in C'$ and a $y \in C$ to a $y' \in C'$, respectively, such that one (x, y)-section of C has length at least $2l/3$ and one (x', y')-section of C' has length at least $l^*/2$. Combining these sections and the two connecting paths, we obtain a cycle of length at least $2l/3 + l^*/2$. By the maximality of C, $l \geqq 3l^*/2$, which again contradicts (4).

Remark 1. The methods described here may also be used to obtain analogous results about 3-connected graphs with prescribed maximum degree d, where $d > 3$. A similar construction, starting with the complete bipartite graph $K_{3,d}$, yield an upper bound of $n^{(\log 2)/\log(d-1)}$. And a corresponding lower bound can be derived by modifying the proof of the theorem so that $p(x)$, $q(x)$ and $r(x)$ are defined to be the terminal edges of the paths $P(x)$, $Q(x)$ and $R(x)$, rather than the terminal vertices.

Remark 2. The point in the above proof where the hypothesis of 3-connectedness (as opposed to 2-connectedness) is crucial is in the assertion that there are three (rather than just two) disjoint paths connecting the disjoint cycles C and C'. This enables one to create a longer cycle than C when C' is at least two-thirds as long as C.

We conjecture that the lower bound can be improved considerably.

Conjecture. There exists a constant $c > 0$ such that $f(n) > n^c$.

References

1. D. Barnette, *Trees in polyhedral graphs*, Can. J. Math. *18* (1966), 731–736.

2. J. A. Bondy and R. C. Entringer, *Longest cycles in 2-connected graphs with prescribed maximum degree*, Can. J. Math., to appear.

3. B. Grünbaum and T. S. Motzkin, *Longest simple paths in polyhedral graphs*, J. London Math. Soc. *37* (1962), 152–160.

4. B. Grünbaum and H. Walther, *Shortness exponents of families of graphs*, J. Combinatorial Theory Ser. A. *14* (1973), 364–385.

5. R. Lang and H. Walther, *Über längste Kreise in regulären Graphen*, Beitrage zur Graphentheorie (Kolloquium, Manebach, 1967). Teubner, Leipzig (1968), 91–98.

6. P. G. Tait, *Remarks on colouring of maps*, Proc. Royal Soc. Edinburgh Ser. A. *10* (1880), 729.

7. W. T. Tutte, *On hamiltonian circuits*, J. London Math. Soc. *21* (1946), 98–101.

8. H. Walther, *Über die Anzahl der Knotenpunkte eines längsten Kreises in planaren, kubischen dreifach knotenzusammenhängenden Graphen*, Studia Sci. Math. Hungar. *2* (1967), 391–398.

9. ——— *Über Extremalkreise in regulären Landkarten*, Wiss. Z. Techn. Hochsch. Ilmenau *15* (1969), 139–142.

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