Annals of Discrete Mathematics 6 (1980) 301–313


# INTERSECTION THEOREMS ON STRUCTURES

Miklós SIMONOVITS and Vera T. SÓS
*Eötvös Loránd University, Budapest, Hungary*

## 1. Introduction

The basic topics of this survey paper are intersection theorems on graphs and on integers. Before turning to these topics let us mention a few things in connection with combinatorial intersection theorems.

The most elementary intersection theorems are the ones on sets. In these cases we fix a set $S$ and a family $\mathcal{A} = \{A_1, \ldots, A_N\}$ of subsets of $S$, and assume that the sets $A_1, \ldots, A_N$ have some intersection property $P$. Then we ask for the maximum $N$ in terms of $|S|$ or other parameters, specified in $P$. More generally instead of an intersection property one can consider any Boole-algebraic property (involving intersection, union, disjointness, complement, containment, rank or size) and ask for maximal or minimal sized families of subsets satisfying the given conditions.

Perhaps the first intersection-type theorem is the Fisher-inequality [11], [14] (which originated from a statistical-block-design-problem) and asserts — in a dual form — that if $A_1, \ldots, A_N$ are $k$-element subsets of a given $n$-element set and $|A_i \cap A_j| = \lambda$ for $1 \leq i < j \leq N$, then $N \leq n$.

Later de Bruijn and Erdős [2] for $\lambda = 1$ and Ryser for arbitrary $\lambda$ proved the following

**Theorem.** *Let $\mathcal{A} = \{A_1, \ldots, A_N\}$ be a family of subsets of an $n$-element set $S$. If*

$$|A_i \cap A_j| = \lambda \quad \textit{for} \quad 1 \leq i < j \leq N,$$

*then $N \leq n$.*

(The difference between this theorem and the Fisher inequality is that the condition $|A_i| = k$ is dropped.)

The extremal systems are known only for $\lambda = 1$ and are isomorphic to one of the following three:

(a) $\{1\}, \{1, 2\}, \ldots, \{1, n\}$,

(b) $\{2, \ldots, n\}, \{1, 2\}, \ldots, \{1, n\}$,

(c) the lines of a finite geometry on $n$ elements when such a geometry exists at all.

Another old, well-known intersection theorem (published only in 1961) is the Erdős–Ko–Rado theorem [8]:

**Theorem.** *Let* $1 \leq l \leq \frac{1}{2}n$. *If* $\mathcal{A} = \{A_1, \ldots, A_N\}$ *is a family of subsets of an* $n$-*element set* $S$ *and* $|A_i| = l$ $(1 \leq i \leq N)$, $A_i \cap A_j \neq \emptyset$ $(1 \leq i < i \leq N)$, *then*

$$N \leq \binom{n-1}{l-1}. \tag{1}$$

Again, one can easily see that this theorem is sharp: Fixing an $x_0 \in S$ and taking all the $l$-element subsets $A_1, \ldots, A_N$ containing $x_0$ $(N = \binom{n-1}{l-1})$ we get a system such that $x_0 \in A_i \cap A_j$ $(i \neq j)$ and hence (1) is sharp. A more general question is the following one (see [23]). Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two given sets of integers and $A_1, \ldots, A_N$ be subsets of $\{1, \ldots, n\}$ satisfying

(a) $\quad |A_i \cap A_j| \in \mathcal{L}_1$ for $1 \leq i < j \leq N$,

(b) $\quad |A_i| \in \mathcal{L}_2$ for $1 \leq i \leq N$.

How large can $N$ be under this condition (for fixed $\mathcal{L}_1$, $\mathcal{L}_2$ and $n$)? This question has been widely investigated recently and many interesting results have been proved in this area e.g. by Ray-Chaudhuri and Wilson [17], Deza and Frankl [6], Deza and Singhi [5].

The purpose of this paper is not to give a survey on intersection theorems on sets. The two theorems above are just two important examples. The reader interested in the details is referred e.g. to the survey papers of Erdős and Kleitman [9], Katona [15] or the "open problem" paper [7] of Erdős. We mention only one more well-known theorem thich is also a characteristic and initiating theorem of this field; Sperner's Theorem [24].

**Theorem.** *Let* $|S| = n$ *and* $A_i \subset S$ *for* $1 \leq i \leq N$ *be such that* $A_i \not\subseteq A_j$ *for* $1 \leq i < j \leq N$. *Then*

$$N \leq \binom{n}{[\frac{1}{2}n]}.$$

This theorem is sharp: the family of subsets of $[\frac{1}{2}n]$ elements of $S$ satisfy the condition $A_i \not\subseteq A_j$ and their number is

$$\binom{n}{[\frac{1}{2}n]}.$$

The above examples concerned two elements of the set-system. This is not necessarily so. A typical result concerning more than two $A_i$'s is the following (Erdős and Kleitman [10]):

Let $A_1, \ldots, A_N$ be a family of subsets of an $n$-element set $S$ and assume that

$A_i \cup A_j = A_k$, $A_i \cap A_j = A_l$ is excluded (except if $i=j=k=l$). Then

$$N = O\left(\frac{2^n}{n^{1/4}}\right)$$

and this estimate is sharp.

Here we give a brief survey on intersection theorems of structural type, where the structures will be either graphs or subsets of integers and the intersection properties will be given in graph theoretical or arithmetical terms.

Before turning to these questions in details we remark that another type of intersection theorems on structures was considered by Deza and Frankl [6], namely, "intersection theorems" on permutations.

The basic problem considered by them is the following one:

**Problem.** Let $P_1, \dots, P_N$ be permutations on $\{1, 2, \dots, n\}$. The distance of $P_i$ and $P_j$: $d(P_i, P_j)$ is the number of non fixed elements of $P_i^{-1}P_j$, that is, the number of elements on which $P_i$ and $P_j$ act differently. How large can $N$ be if

(a) $\quad d(P_i, P_j) \leq \lambda \quad (1 \leq i < j \leq N)$,

or if

(b) $\quad d(P_i, P_j) \geq \lambda \quad (1 \leq i < j \leq N)$.

Some of their results were published in [6], some others can be found in this volume.

## 2. Graph intersection problems, introductory examples

In this chapter we shall consider graphs without loops or multiple edges.

**Definition.** If $G$ and $H$ are graphs on the same vertex set $V$, their intersection $G \cap H$ is the graph whose edge-set $E(G \cap H) = E(G) \cap E(H)$ and whose vertices are the elements of $V$ incident to some edges in $E(G \cap H)$.

Given a family $\mathcal{L}$ of graphs, $f(n; \mathcal{L})$ is the maximum number of graphs $G_1, \dots, G_N$ defined on the same $n$-element vertex-set $V$ for which

$$G_i \cap G_j \in \mathcal{L} \quad (1 \leq i < j \leq N).$$

**Remark 2.1.** Observe that $G_i \cap G_j$ has no isolated vertices.

Below we shall consider some special cases of the general problem. In Example 2.3 $\mathcal{L}$ is the family of stars and $f(n, \mathcal{L}) = 2^{n-1}$, in Theorem 4.1 $\mathcal{L}$ is the family of paths and $f(n, \mathcal{L}) = o(n^5)$, that is, much smaller. One could ask: both the path and the star are trees, why is $f(n, \mathcal{L})$ "very large" in the first case and "very small" in the second one. This is, how we arrived at the following non-trivial (and perhaps too general?) problem.

**Problem 2.1.** Under which conditions on $\mathcal{L}$ is $f(n, \mathcal{L})$ polynomially bounded (i.e., when do we know that there is an $r$ such that $f(n, \mathcal{L}) = o(n^r)$?

**Theorem 2.1.** *Let $\mathcal{L}$ be the family of graphs isomorphic to some of $k$ given graphs $L_1, \dots, L_k$. Then*

$$f(n, \mathcal{L}) = o(n^{v+2})$$

*for $v = \max_{1 \le i \le k} v(L_i)$ (where $v(G)$ denotes the number of vertices of $G$).*

Theorem 2.1 is very easy to prove. Its main content is that for finite families $\mathcal{L}$ $f(n, \mathcal{L})$ is always polynomially bounded. From now on we shall restrict our investigations to infinite $\mathcal{L}$'s.

**Theorem 2.2** [19]. *Let $\mathcal{L}$ be a family of graphs with minimum valency $\ge 2$ and maximum valency $\le K$, for which the number of components and the number of vertices of valency $\ne 2$ are also $\le K$. There exists an $r = r_K$ such that*

$$f(n, \mathcal{L}) = o(n^r).$$

**Remark.** Another way of formulating the condition of this theorem is to assume the existence of a finite family $\{L_1, \dots, L_k\}$ of graphs with minimum valence $\ge 2$ such that the family of graphs topologically equivalent with some of $L_1, \dots, L_k$ forms $\mathcal{L}$. By the way, Theorem 2.2 is sharp in more than one way.

(a) The condition on the boundedness of the number of components in $\mathcal{L}$ is necessary: let $\mathcal{L}$ be the family of graphs consisting of vertex disjoint $K_3$'s. All the other conditions of Theorem 2.2 are satisfied, still $f(n, \mathcal{L}) \ge 2^{[n/3]}$. Indeed, let us fix $[\frac{1}{3}n]$ vertex-disjoint triangles on an $n$-element set $V$ and let $G_1, \dots, G_N$ ($N = 2^{[n/3]}$) be the family of all graphs on $V$ containing all the edges of some of the considered triangles and none edges of the other ones. Clearly, $G_i \cap G_j \in \mathcal{L}$.

(b) The condition on the maximum valency is also necessary: let $K_2(p; q)$ denote the complete bipartite graph of $p$ and $q \ge p$ vertice in its classes. Let $\mathcal{L} = \{K_2(2, q); q = 2, 3, 4, \dots\}$. We assert that

$$f(n; \mathcal{L}) \ge 2^{n-4}.$$

Indeed, let $V = \{1, \dots, n\}$ and $G_1, \dots, G_N$ ($N = 2^{n-4}$) be the family of $K_2(2, q)$'s the first class of which is $\{1, 2\}$, the second one is in $\{3, \dots, n\}$ and it contains $\{3, 4\}$. Clearly, we have $2^{n-4}$ such graphs and $G_i \cap G_j \in \mathcal{L}$. Thus the condition on the maximum valency is really necessary.

(c) The condition on the number of vertices of valence $\ne 2$ is also necessary: let $\mathcal{L}$ be the family for all the graphs which have only valences 2 and 3.

Let $G$ be a cubic graph on $n$ vertices with a one-factor $P$. For each $P' \subseteq P$ we form $G_{P'} =: G - P'$. Any two of these $2^{[n/2]}$ graphs intersect in a graph $\in \mathscr{L}$. Thus the bound on the number of vertices of valence $\neq 2$ is also needed.

The examples above show not only that Theorem 2.2 is sharp. They show that $f(n, \mathscr{L})$ is very often exponentially large. We give a few more illustrations of that.

**Example 2.1.** Let $\mathscr{L}$ be the family of all complete graphs, $K_0 = \emptyset$ *included*. It is easy to see that

$$f(n, \mathscr{L}) = 2^n \quad (n > n_0).$$

**Example 2.2.** Let $\mathscr{L}$ be the family of all complete graphs, $K_0 = \emptyset$ *excluded*. Obviously,

$$f(n, \mathscr{L}) \geq 2^{n-2}$$

(which can be achieved e.g. by fixing 2 vertices $x$, $y$ and taking all the complete graphs containing $(x, y)$).

**Example 2.3.** Let $\mathscr{L}$ be the family of stars:

$$\mathscr{L} = \{K_2(1, q): q = 0, 1, 2, \ldots\}.$$

Clearly,

$$f(n, \mathscr{L}) = 2^{n-1} \quad (n \geq n_0)$$

and the only extremal system can be described as follows: We fix a $G = K_2(1, n-1)$ and each $G_i$ $(i = 1, \ldots, N)$ has the same $n$ vertices as $G$ and some edges of $G$. This example is very similar to that of (b) in the above remark, still it is important since it shows that if we assume that the intersection of any two graphs is a *tree*, the extremal system will be exponentially large.

**Example 2.4.** Let $\mathscr{L}$ be the family of all the connected graphs. Then

$$f(n, \mathscr{L}) \geq 2^{\binom{n-1}{2}}.$$

To show this we fix a tree $T$ on $V = \{1, \ldots, n\}$ and consider all the graphs $G_i \supseteq T$. This immediately proves (2.1). However, (2.1) is not the best. To improve it we consider the following family $G_1, \ldots, G_N$. $V = \{1, \ldots, n\}$ is fixed and we choose an arbitrary graph $G^{n-1}$ with $V(G^{n-1}) = \{1, \ldots, n-1\}$. Then for each edge $(i, j)$ of $G^{n-1}$ we join max $(i, j)$ to $n$ by an edge. Finally we add an arbitrary number of edges $(i, n)$ to this graph. Let $G_1, \ldots, G_N$ be the family of all the graphs obtained in this way. Obviously $G_i \cap G_j$ is always connected. It is not difficult to compute that

$$N \geq (4.6)2^{\binom{n-1}{2}}.$$

**Open problem.** Let $\mathscr{L}$ be the family of connected graphs. Is

$$f(n, \mathscr{L}) = \mathrm{O}(2^{\binom{n-1}{2}})?$$

If not, what is the proper order of magnitude of $f(n, \mathscr{L})$?

## 3. Intersection theorems on graphs where the intersections are cycles

In this and the next chapters we try to give a description of the cases, when $\mathscr{L}$ consists of cycles or paths. Let $\mathscr{A}_1$ resp. $\mathscr{A}_2$ be the family of all cycles, $\emptyset$ included in $\mathscr{A}_1$ but excluded from $\mathscr{A}_2$.

**Definition.** Given an $\mathscr{L}$, the family $G_1, \ldots, G_N$ will be called an $\mathscr{L}$-intersection family if $G_i \cap G_j \in \mathscr{L}$ whenever $1 \leq i < j \leq N$.

**Definition.** A family $G_1, \ldots, G_N$ is a strong $\Delta$-system if

$$G_i \cap G_j = \bigcap_{l \leq N} G_l \quad 1 \leq i < j \leq N$$

i.e. $G_i \cap G_j$, is independent of $i$ and $j$. $\bigcap_{i \leq N} G_i = K$ is called the kernel.

In [19] we proved

**Theorem 3.1.** *If $n \geq 4$, then $f(n, \mathscr{A}_2) = \binom{n}{2} - 2$ and the only extremal system, that is, the only $\mathscr{A}_2$–intersection system $G_1, \ldots, G_N$ for $N = f(n, \mathscr{A}_2)$ is the following one: $E(G_1)$ form a triangle and $E(G_i)$ contains $E(G_1)$ and exactly one additional edge for $i = 2, \ldots, \binom{n}{2} - 2$.*

It is worth noticing, that the extremal system of Theorem 3.1 is a strong $\Delta$–system. We also prove the following theorem:

**Theorem 3.2.** *If $G_1, \ldots, G_N$ form an $\mathscr{A}_2$-intersection system on $n$ vertices but not a strong $\Delta$-system, then*

$$N \leq \frac{1}{\sqrt{6}} n^2 + n.$$

The basic idea behind proving Theorem 3.2 is the following one: this theorem shows, that if a system is almost extremal, say, it has $\geq \binom{n}{2} - \mathrm{o}(n^2)$ graphs, then it is already a strong $\Delta$-system. This means that in some sense the property of being strong $\Delta$-system is an important (or at least a stable) feature of the extremal system.

Actually Theorem 3.1 is a simple consequence of Theorem 3.2. We do not think that Theorem 3.2 is sharp. On the contrary:

**Conjecture.** *Let $G_1, \dots, G_N$ be graphs on $n$ vertices forming an $\mathscr{A}_2$-intersection family but not a strong $\Delta$-system. Then*

$$N \leq \tfrac{1}{6}n^2 + o(n^2).$$

The above conjecture is sharp if true: on $n$ points we can find $s \approx \frac{1}{6}n^2$ triangles $T_1, \dots, T_s$ such that no two of them have a common edge. Let $E(G_i) = E(T_s) \cup E(T_i)$ for $i < s$ and $E(G_m) = \bigcup_{j<s} E(T_j)$. One can easily see that $G_i \cap G_j$ is always a triangle but it does depend on $(i, j)$. Another more symmetric construction of this type can be found in [19].

Our next result concerns the case then $G_i \cap G_j = \emptyset$ is also allowed. In [19] we have proved that:

**Theorem 3.3.** *Let $n \geq 10$, $\binom{x}{2} =: \frac{1}{2}x(x-1)$ for any real $x$ and $s =: \frac{1}{3}\binom{n}{2}$. Then*

$$\binom{s-n}{2} < f(n, \mathscr{A}_1) \leq \binom{s}{2} + s + 1.$$

*The upper bound is the best possible if $n = 6k+1$ or $n = 6k+3$.*

**Remark.** Theorem 3.3 implies that

$$f(n, \mathscr{A}_1) \approx \tfrac{1}{72}n^4.$$

**Remark.** If $n = 6k+1$ or $n = 6k+3$, we can easily describe the only extremal system. We choose a Steiner triple system $T_1, \dots, T_s$ on $\{1, \dots, n\}$ and $\{G_{(i,j)};\ i, j \leq s\}$ are the $\binom{s}{2} + s$ graphs defined by $V(G_{(i,j)}) = \{1, \dots, n\}$ and $E(G_{(i,j)}) = E(T_i) \cup E(T_j)$ where $1 \leq i \leq j \leq s$. (Here $T_i$ is considered as a $K_3$!) Finally we add a graph $G^n$ with no edges.

**Remark.** Independently from us V. Rödl also proved some of the results of [19], which include

$$f(n, \mathscr{A}_1) = o(n^4)$$

and the estimate on $f(n, \mathscr{A}_4)$ from the next chapter (unpublished).

## 4. Intersection problems, where the intersections are paths

The intersection problems of the paths are more difficult than the intersection problems of the cycles and the reason for that is probably, that the union of two cycles is never a cycle, while the union of two paths can easily be a path.

**Notation.** Let $\mathscr{A}_3$ be the family of paths, $\emptyset$ included, while $\mathscr{A}_4$ be the family of nonempty paths.

In [20] we have proved that

**Theorem 4.1.**

$$f(n, \mathscr{A}_3) = \mathrm{o}(n^5),$$
$$f(n, \mathscr{A}_4) = \mathrm{o}(n^4).$$

The following constructions show that (apart from the multiplicative constant) Theorem 4.1 is sharp:

**Construction 4.1.** Let us divide $n$ vertices into five classes $C_1, \ldots, C_5$ of $\approx \frac{1}{5}n$ vertices each. Let us take all the pentagons $(x_1, \ldots, x_5)$ where $x_i \in C_i$ for $1 \leqslant i \leqslant 5$. Obviously, any two pentagons intersect in a path and their number is $\geqslant (\frac{1}{5}n)^5$. To get $G_1, \ldots, G_N$ we add $n-5$ isolated vertices to each of them. Thus we see that

$$f(n, \mathscr{A}_3) \geqslant \frac{1}{5^5} n^5.$$

The multiplicative constant $1/5^5$ is not sharp.

**Open questions.** What is $\overline{\lim}\, f(n, \mathscr{A}_3)/n^5$? Does the limit exist? Is it true that if $G_1, \ldots, G_N$ is an extremal system for $\mathscr{A}_3$, then all but $\mathrm{o}(n^4)$ graphs are pentagons? (If yes, then the above limit does exist.)

**Construction 4.2.** Let the disjoint classes $C_1$, $C_2$, $C_3$, $C_4$ have $\approx \frac{1}{4}(n-k)$ vertices each and take all the paths $P^{k+4}$ of form $(x_1, x_2, u_1, \ldots, u_k, x_3, x_4)$ where $x_i \in C_i$ and $u_1, \ldots, u_k$ are fixed vertices outside $\bigcup C_i$. Obviously, the intersection of any two such paths is a path of $\geqslant k$ vertices. Thus

$$f(n, \mathscr{A}_4) \geqslant \tfrac{1}{64} n^4 - \mathrm{O}(n^3).$$

This construction shows that the second assertion of Theorem 4.1 is also sharp. It also answers the question posed by A. Frankel after our lecture, namely, it shows that if we allow only long paths as intersections, that will not diminish the original $\mathrm{O}(n^4)$. The first two questions asked above in connection with $\mathscr{A}_3$ are also interesting in connection with $\mathscr{A}_4$: what is $\overline{\lim}\; f(n, \mathscr{A}_4)/n^4$ and does the limit exist?

## 5. Intersection properties of subset of integers

Here we consider problems of the following type:
Let $\mathscr{L}$ be a given family of sets of integers and let $\{A_1, \ldots, A_N\}$ be a family of

subsets of $\{1, \ldots, n\}$. How large can $N$ be if

$$A_i \cap A_j \in \mathscr{L} \quad (1 \leq i < j \leq N).$$

The maximum will again be denoted by $f(n, \mathscr{L})$. The simplest case is the case of intervals. The set $\{a, a+1, \ldots, a+k-1\}$ will be called an interval of length $k$ and $I_k$ will denote the family of intervals of length $\geq k$. One can easily see that

$$f(n, I_0) = \binom{n}{2} + n + 1.$$

An extremal system is the family of subsets of $\leq 2$ elements, $\emptyset$ included. (Other extremal systems can be obtained by replacing *some* pairs $\{a_i, b_i\}$ by the corresponding intervals.) Further,

$$f(n, I_1) = [\tfrac{1}{4}(n+1)^2]$$

and an extremal system can be obtained by considering all the intervals containing $m = [\frac{1}{2}(n+1)]$.

These results can be found in the paper of Graham, Simonovits and Sós [13] and the method used there also shows that

$$f(n, I_k) = [\tfrac{1}{4}(n-k+2)^2]$$

if $k \geq 2$, $n \geq k$.

The arithmetic progression-problem is already more difficult. Let $P_k$ denote the family of all arithmetic progressions of $\geq k$ terms. Then

$$f(n, P_0) = \binom{n}{3} + \binom{n}{2} + n + 1.$$

The extremal system is uniquely determined: it consists of all the subsets of $\{1, \ldots, n\}$ of $\leq 3$ elements, [13]. The basic idea to prove this is that if $A_1, \ldots, A_N$ is a family such that $A_i \cap A_j \in P_0$, then the first, second and last elements of an $A_i$ will uniquely determine it.

In some sense the elements of $P_0$ are not what we wanted: in the case of the extremal system $|A_i \cap A_j| \leq 2$ and this is, why it is a (trivial) arithmetic progression.

It is natural to ask, how large $f(n, P_k)$ is for $k \geq 3$, when the fact that $A_i \cap A_j$ is an arithmetic progression, really makes sense. We proved that [21]:

**Theorem 5.1.**

$$f(n, P_k) = (\tfrac{1}{24}\pi^2 + o(1))n^2 \quad (k \geq 2).$$

To show that

$$f(n, P_k) \geq (\tfrac{1}{24}\pi^2 + o(1))n^2$$

we consider all the arithmetic progressions $A_1, \ldots, A_N$ of form

$$A_i = \{[\tfrac{1}{2}n] + jd: \quad j = -a, -a+1, \ldots, -1, 0, 1, \ldots, b\}$$

where $d \leq n^{1/3}$, $\sqrt{n} \leq b \leq n/2d$ and $a \leq (n-1)/2d$. One can easily see that $A_i \cap A_j \in P_m$ for some $m \geq \frac{1}{3}n^{1/6}$ and

$$N = \tfrac{1}{4}n^2\left(\sum_{d=1}^{\infty} \frac{1}{d^2} + o(1)\right) = (\tfrac{1}{24}\pi^2 + o(1))n^2.$$

The upper part is much deeper. Actually we deduced Theorem 5.1 from Theorem 5.2:

**Theorem 5.2.** *Let $k \geq 2$ be fixed and $A_1, \ldots, A_N$ be subsets of $\{1, \ldots, n\}$ such that $A_i \cap A_j \in P_k$ for $1 \leq i < j \leq N$. Let us assume that none of the $A_i$'s is an arithmetic progression. Then*

$$N = o(n^{5/3} \log^3 n). \tag{5.1}$$

**Remarks.** (i) Obviously Theorem 5.2 implies Theorem 5.1, since the number of arithmetic progressions is only $(\frac{1}{24}\pi^2 + o(1))n^2$. It also shows that all the almost extremal systems of Theorem 5.1 consist primarily of arithmetic progressions.

(ii) We can improve the exponent of $\log n$ in (5.1) but we do not know, whether $n^{5/3}$ can be replaced by some smaller power of $n$.

Though a one-term arithmetic progression is not a real arithmetic progression, we wanted to determine $f(n, P_1)$ as well.

**Conjecture.** $f(n, P_1) = \binom{n}{2} + 1$ *and one extremal system $\{A_1, \ldots, A_N\}$ is the system of sets of form $\{c, x, y\}$ for some fixed $c$ (where $x = y$ or $x = c$ or $x = y = c$ are also allowed).*

There are also other, slightly different extremal systems if the conjecture is true. We could not prove it, but we can prove e.g. that

$$f(n, P_1) \leq \binom{n-1}{2} + \tfrac{1}{24}\pi^2 n^2 + o(n \log n).$$

**Open problem.** Let $A_1, \ldots, A_N$ be a extremal system in Theorem 5.1, i.e. for $P_k$ $(k \geq 2)$. Is it true that all the sets $A_i$ are arithmetic progressions?

## 6. Connections between Ramsey theory and graph intersection problems

As we mentioned and tried to "prove by some examples", the theory of intersection theorems of sets or other structures is a very wide and rapidly growing field of investigation. It has also many interesting, sometimes fairly deep connections to some other fields of mathematics. Obviously, the theory of block designs is such a field, [23, 14]. Some extremal graph problems, or combinatorial problems of finite geometries are also very strongly connected to intersection problems on sets [23]. An extensive part of hypergraph theory is nothing but intersection theorems in another language. (Thus e.g. the survey paper of Katona on intersection theorems has the title "Extremal problems for hypergraphs" [15].) Many questions in combinatorial number theory finally boil down to intersection theorems.

In this paragraph we have a closer look at the connection between the intersection theorems and another rapidly growing field of combinatorics, namely the (finite) Ramsey theory. We restrict ourselves to the case of ordinary graphs and 2 colours. It is well-known, that there exist two constants $c_1>0$ and $c_2>0$ such that if we colour the edges of a $K_n$ by two colours, say red and blue, then $K_n$ contains either a red or a blue (i.e. monochromatic) $K_m$ for $m=[c_1 \log n]$. On the other hand, it is easy to see that if we colour the edges of $K_n$ in a random way, i.e. colouring each edge of $K_n$ by red with probability $\frac{1}{2}$, independently from the colouring of the other edges, then the largest monochromatic $K_m \subseteq K_n$ will have size $m \leqslant [c_2 \log n]$ with probability tending to 1 (as $n \to \infty$). This means that we can show by probabilistic methods the sharpness of the lower bound $m=[c_1 \log n]$ in this Ramsey problem. It is a longstanding and famous unsolved problem, how can one give a constructive upper bound:

Let us construct a two-colouring of the edges of $K_n$ such that the largest monochromatic $K_m$ has $m \leqslant [c_3 \log n]$ vertices. One can construct easily a 2-colouring with $m=[\sqrt{n}]+1$: the $n$ vertices are divided into $\{\sqrt{n}\}$ classes as uniformly as possible and $(x, y)$ is blue or red depending on whether they belong to the same class or not.

The sharper $m \sim n^{1/3}$ was done by Zs. Nagy and the basic idea was to use two simple intersection theorems. The vertices of $G^n$ are the triples of a $k \sim (6n)^{1/3}$ set and $\{a, b, c\}$ is joined by a red edge to $\{a', b', c'\}$ iff $|\{a, b, c\} \cap \{a', b', c'\}|$ is even. It is almost trivial that the largest monochromatic $K$ has $m=O(k)=O(n^{1/3})$ vertices in this colouring.

The next step was due to Abbott's 1, who proved that if one has a fixed graph $G^n$ with the maximum $m=n^{\varepsilon}$, then one can very easily construct infinitely many such graphs. Abbott's construction combined with Erdős's probabilistic method yields $m=n^{\varepsilon}$ in a semiconstructive way, also "showing" that the notion of constructiveness is not quite well-defined in combinatorics.

The latest development for this question is the following.

Frankl [12] constructed for $h=2k^2$ two sets of integers

$$\mathscr{L}_1=\{2ak+b \mid 0 \leqslant a, b \leqslant k-1\}$$

and

$$\mathscr{L}_2=\{0, \ldots, h-1\}-\mathscr{L}_1$$

so that if we take all the $2k^2$-element set of an $n$-element set and join two of them, say $A_i$ and $A_j$, by a red edge if $|A_i \cap A_j| \in \mathscr{L}_1$ and by a blue edge otherwise (i.e. if $|A_i \cap A_j| \in \mathscr{L}_2$), then, using two appropriate intersection theorems, he could prove that the largest monochromatic $K_m$ has $m \leqslant cn^{1/k}$ vertices in this colouring.

Obviously, the basic idea in the above construction is that $f(n, \mathscr{L}_1)$, $f(n, \mathscr{L}_2)$ are much smaller than $f(n, \mathscr{L}_1 \cup \mathscr{L}_2) = 2^n$. We arrived originally to the same problem for graphs from the other direction:

**Open problem.** Given two families of graphs: $\mathscr{L}_1$ and $\mathscr{L}_2$, is it true that if $f(n, \mathscr{L}_1)$ and $f(n, \mathscr{L}_2)$ are polynomially bounded, then $f(n, \mathscr{L}_1 \cup \mathscr{L}_2)$ is also polynomially bounded?

(If the answer is "yes", that is interesting in itself, but if the answer is "no", then one could hope for a sharpening of the Frankl construction. Further, if in the Ramsey theorem we had only say $n^\varepsilon$ instead of $c \log n$ (for some fixed $\varepsilon > 0$) which is definitely not the case, that would imply the solution of the above problem.)

## 7. Some further open problems

The basic problem, we are interested in in connection with graph intersection problem is, as we stated, on which properties of $\mathscr{L}$ does the fact depend, whether $f(n, \mathscr{L})$ is polynomially bounded or not. The first two problems below may seem very special, we picked them because we feel that if one can solve them, one gets a step nearer to the solution of the main problem.

**Problem 7.1.** Let $L_k$ denote the graph on $k^2$ vertices whose edges form $k$ independent paths of $k$ vertices (each). Let $\mathscr{L} = \{L_3, \ldots\}$. Is $f(n, \mathscr{L})$ polynomially bounded?

**Problem 7.2.** What type of "graph products" produces "polynomial families" from "polynomial families". E.g. let $L_k = P^k \times P^k$ be the graph whose vertices are $(i, j)$ $1 \leq i \leq k$; $1 \leq j \leq k$ and $(i, j)$ is joined to $(i', j')$ if $|i - i'| = 1$ and $j = j'$ or $|j - j'| = 1$ and $i = i'$. ($L_k$ is a "square subdivided into smaller squares".) Is $f(n, \{L_2, \ldots\})$ polynomially bounded?

The next problem concerns trees, where it is obvious that $f(n, \mathscr{L})$ is exponentially large (see Section 3.).

**Problem 7.3.** Let $\mathscr{L}$ be now the family of all the trees. How large $f(n, \mathscr{L})$ can be?

(Here we would like to have an asymptotically sharp estimate of $\log f(n, \mathscr{L})$.) Finally we formulate a general problem.

**Problem 7.4.** Is it true that if $f(n, \mathscr{L})$ is polynomially bounded for a family of graphs, then there exists an *integer* $r$ such that $\lim_{n\to\infty} f(n, \mathscr{L})/n^r$ exists and is *positive*?

## Note added in proof

(1) The Open Question in Section 4 was solved in the affirmative by Z. Füredi.

(2) V. Rödl has published his results in Comment Math. Univ. Carolinae 19 (1) (1978) 135–140.

## References


[1] H.L. Abbott, Lower bounds for some Ramsey numbers, Discrete Math. 2 (1972) 289–294.

[2] N.G. de Bruijn and P. Erdős, On a combinatorial problem, Nederl. Akad. Wetensch. Proc. 51 (1948) 1277–1279.

[3] M. Deza, On maximal sets of permutations with given maximal distance, to appear.

[4] M. Deza, P. Erdős and P. Frankl, Intersection properties of systems of finite sets, Proc. London Math. Soc. 36 (1978) 369–384.

[5] M. Deza, P. Erdős and N.M. Singhi, Combinatorial problems on subsets and their intersections, Advances in Math.

[6] M. Deza and P. Frankl, On the maximum number of permutations with given maximal or minimal distance, J. Combin. Theory 22(A) (1977) 352–360.

[7] P. Erdős, Problems and results in combinatorial analysis, in: Proc. Coll. Int. Sulle Teorie Combinatorie, Roma, 1973, Accad. Nazionale dei Lincei (1973).

[8] P. Erdős, Chao Ko and R. Rado, Intersection theorems for systems of finite sets, Quart. J. Math. Oxford Ser. 12 (2) (1961) 313–318.

[9] P. Erdős and D.J. Kleitman, Extremal problems among subsets of a set, in: Combinatorial Mathematics and Applications, Proc. Chapel Hill Conference (1970)146–170.

[10] P. Erdős and D.J. Kleitman, On collections of subsets containing no 4-member Boolean algebra, Proc. Am. Math. Soc. 28 (1971) 507–510.

[11] R.A. Fisher, An examination of the different possible solutions of a problem in incomplete blocks, Ann. Eugenics 10 (1940) 52–75.

[12] P. Frankl, A constructive lower bound for some Ramsey numbers, Ars Combinatoria 3 (1977) 297–302.

[13] R.L. Graham, M. Simonovits and V.T. Sós, A note on the intersection properties of subsets of integers, to appear.

[14] M. Hall, Combinatorial Theory (Blaisdell Publ. Company, 1967).

[15] G. Katona, Extremal problems for hypergraphs, in: M. Hall and Von Lindt (eds), Combinatorics, Vol. 2, Math. Centre Tracts, 56 (Math. Centre, Amsterdam, 1974) 13–42.

[16] D. Lubell, A short proof of Sperner's lemma, J. Combin. Theory 1 (1966) 299.

[17] D.K. Ray-Chaudhuri and R.M. Wilson, On $t$-designs, Osaka J. Math. 12 (1975) 737–744.

[18] H.J. Ryser, On extension of a theorem of De Bruijn and Erdős on combinatorial designs, J. Algebra 10 246–261.

[19] M. Simonovits and V.T. Sós, Intersection theorems for graphs II, Coll. Math. Soc. J. Bolyai 18 Combinatorics, Keszthely, Hungary (1976) 1017–1029.

[20] M. Simonovits and V.T. Sós, Intersection theorems for graphs, Coll. Internationaux C.N,R.S. No. 260, Problemes Combinatoires et Theorie des Graphes, 389–391.

[21] M. Simonovits and V.T. Sós, Intersection properties of subsets of integer, to appear.

[22] V.T. Sós, Some remarks on the connection of graph theory, finite geometry and block designs, in: Proc. Coll. Internazionale sulle Teorie Combinatorie, Roma, 1973, Accademia Nazionale dei Lincei (1973) 223–233.

[23] E. Sperner, Ein Satz über Untermenge einer endlichen Menge, Math. Z. 27 (1928) 544–548.