SIAM J. DISCRETE MATH.
Vol. 31, No. 2, pp. 1017–1071

THE APPROXIMATE LOEBL–KOMLÓS–SÓS CONJECTURE III: THE FINER STRUCTURE OF LKS GRAPHS*

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Abstract. This is the third of a series of four papers in which we prove the following relaxation of the Loebl–Komlós–Sós conjecture: For every $\alpha > 0$ there exists a number k_0 such that for every $k > k_0$, every n-vertex graph G with at least $(\frac{1}{2}+\alpha)n$ vertices of degree at least $(1+\alpha)k$ contains each tree T of order k as a subgraph. In the first paper of the series, we gave a decomposition of the graph G into several parts of different characteristics. In the second paper, we found a combinatorial structure inside the decomposition. In this paper, we will give a refinement of this structure. In the fourth paper, the refined structure will be used for embedding the tree T.

Key words. extremal graph theory, Loebl–Komlós–Sós conjecture, tree embedding, regularity lemma, sparse graph, graph decomposition

AMS subject classifications. Primary, 05C35; Secondary, 05C05

DOI. 10.1137/140982866

1. Introduction. This is the third of a series of four papers [HKP+a, HKP+b, HKP+c, HKP+d] in which we provide an approximate solution of the Loebl–Komlós–Sós conjecture. The conjecture reads as follows.

CONJECTURE 1.1 (Loebl–Komlós–Sós conjecture 1995 [EFLS95]). *Suppose that G is an n-vertex graph with at least $n/2$ vertices of degree more than $k-2$. Then G*

*Received by the editors August 18, 2014; accepted for publication (in revised form) January 31, 2017; published electronically May 25, 2017. A condensed description of this work appeared in [HPS+15].

http://www.siam.org/journals/sidma/31-2/98286.html

Funding: The first author's research leading to these results received funding from the People Programme (Marie Curie Actions) of the European Union's Seventh Framework Programme (FP7/2007-2013) under REA grant 628974. Much of the work was done while the first author was supported by an EPSRC postdoctoral fellowship EP/I026630/1 while affiliated with DIMAP and the Mathematics Institute, University of Warwick. The Institute of Computer Science of the Czech Academy of Sciences is supported by RVO:67985807. The third author was supported by the Marie Curie fellowship FIST, DFG grant TA 309/2-1, Czech Ministry of Education project 1M0545, EPSRC award EP/D063191/1, and EPSRC Additional Sponsorship EP/J501414/1. The research leading to these results received funding from the European Union Seventh Framework Programme (FP7/2007-2013) under grant PIEF-GA-2009-253925. The work leading to this invention was supported by the European Regional Development Fund (ERDF), project "NTIS – New Technologies for the Information Society," European Centre of Excellence, CZ.1.05/1.1.00/02.0090. The third author was partially supported by the Czech Science Foundation, grant GJ16-07822Y. The fourth author was supported by OTKA 78439, OTKA 101536, OTKA 116769, and ERC-AdG. 321104. The fifth author was supported by Fondecyt Iniciacion grant 11090141, Fondecyt Regular grant 1140766, CMM Basal, and Nucleo Milenio Información y Coordinación en Redes ICM/FIC P10-024F. The sixth author was supported by OTKA 104483, OTKA 101536, and ERC-AdG. 321104.

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contains each tree of order k.

We discuss the history and state of the art in detail in the first paper [HKP$^+$a] of our series. The main result, which will be proved in [HKP$^+$d], is the approximate solution of the Loebl–Komlós–Sós conjecture, namely the following.

THEOREM 1.2 (main result [HKP$^+$d]). *For every $\alpha > 0$ there exists k_0 such that for any $k > k_0$ we have the following: Each n-vertex graph G with at least $(\frac{1}{2} + \alpha)n$ vertices of degree at least $(1 + \alpha)k$ contains each tree T of order k.*

In the first paper [HKP$^+$a], we exposed the decomposition techniques, finding a *sparse decomposition* of the host graph G. The sparse decomposition should be thought of as a counterpart to the Szemerédi regularity lemma (but compared to the Szemerédi regularity lemma, the sparse decomposition seems to be less versatile). In the second paper [HKP$^+$b], we combined the sparse decomposition with a matching structure, obtaining in [HKP$^+$b, Lemma 5.4] what we call *the rough structure*. The rough structure obtained in [HKP$^+$b, Lemma 5.4] depends on the graph G only, i.e., is independent of the tree T. The rough structure encodes the general information on how T should be embedded on a macroscopic scale. However, from the perspective of embedding small parts of T locally, the properties of the rough structure are insufficient. In the present paper we take the preparation of the host graph one step further, refining the rough structure. This way we obtain one of ten possible *configurations*. Formally, each of the configurations—denoted by $(\diamond\mathbf{1})$–$(\diamond\mathbf{10})$—is a collection of favorable properties the host graph must satisfy. Each of these configurations is based on the building blocks of the sparse decomposition and describes in a very fine way a substructure in G. Some of the configurations involve some basic parameters of the tree T. That is, while the presence of some individual configurations (namely, configurations $(\diamond\mathbf{1})$–$(\diamond\mathbf{5})$ and $(\diamond\mathbf{10})$ introduced in section 3) suffices for embedding of each k-vertex tree, configurations $(\diamond\mathbf{6})$–$(\diamond\mathbf{9})$ are accompanied by parameters (denoted by h, h_1, and h_2 in Definitions 4.11–4.14) that depend on certain parameters of the tree T.

In the final paper [HKP$^+$d], we will prove that each of these ten configurations allows us to embed T. This will complete the proof of Theorem 1.2. An overview of how the embedding goes for each individual configuration is given in [HKP$^+$d, section 6.1]. We recommend that the reader consult this part of [HKP$^+$d] in parallel when reading through the definitions of the configurations in section 4.

The paper is organized as follows. In section 2, we introduce some basic notation. In section 3, we introduce some further auxiliary notions and two "settings" that will be common to the rest of the paper. In section 4, we present the main result of this paper, Lemma 4.17. The lemma states that in any graph that satisfies the conditions of Theorem 1.2, we can find at least one of the ten configurations described above. To prove it, we first introduce some preliminary "cleaning lemmas" in section 5. The proof of Lemma 4.17 then occupies section 6. This is illustrated in Figure 1.

2. Notation, basic facts, and bits from other papers in the series.

2.1. General notation. The set $\{1, 2, \ldots, n\}$ of the first n positive integers is denoted by $[n]$. We frequently employ indexing by many indices. We write superscript indices in parentheses (such as $a^{(3)}$), as opposed to notation of powers (such as a^3). We sometimes use subscripts to refer to parameters appearing in a fact/lemma/theorem. For example, $\alpha_{\mathrm{T}1.2}$ refers to the parameter α from Theorem 1.2. We omit rounding symbols when this does not affect the correctness of the arguments.

Table 1 shows the system of notation that we use in this paper and in [HKP$^+$a, HKP$^+$b, HKP$^+$d].

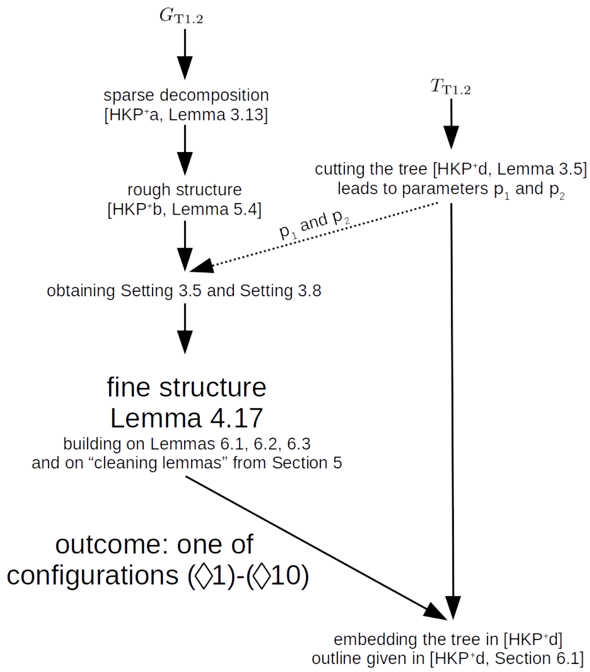


FIG. 1. *Diagram of the proof of Theorem* 1.2 *with focus on the part dealt with in this paper.*

TABLE 1
Specific notation used in the series.

lower case Greek letters	small positive constants ($\ll 1$) ϕ reserved for embedding; $\phi : V(T) \to V(G)$
upper case Greek letters	large positive constants ($\gg 1$)
one-letter bold	sets of clusters
bold (e.g., $\mathbf{trees}(k), \mathbf{LKS}(n,k,\eta)$)	classes of graphs
blackboard bold (e.g., $\mathbb{H}, \mathbb{E}, \mathbb{S}_{\eta,k}(G), \mathbb{XA}$)	distinguished vertex sets except for $\mathbb{N}$, which denotes the set $\{1,2,\dots\}$
calligraphic (e.g., $\mathcal{A}, \mathcal{D}, \mathcal{N}$)	families (of vertex sets, "dense spots," and regular pairs)
∇(=nabla)	sparse decomposition (see Definition 2.11)

We write $V(G)$ and $E(G)$ for the vertex set and edge set of a graph G, respectively. Further, $v(G) = |V(G)|$ is the order of G, and $e(G) = |E(G)|$ is its number of edges. If $X, Y \subseteq V(G)$ are two, not necessarily disjoint, sets of vertices, we write $e(X)$ for the number of edges induced by X, and $e(X, Y)$ for the number of ordered pairs $(x, y) \in X \times Y$ such that $xy \in E(G)$. In particular, note that $2e(X) = e(X, X)$.

For a graph G, a vertex $v \in V(G)$, and a set $U \subseteq V(G)$, we write $\deg(v)$ and $\deg(v, U)$ for the degree of v and for the number of neighbors of v in U, respectively. We write $\mathrm{mindeg}(G)$ for the minimum degree of G, $\mathrm{mindeg}(U) := \min\{\deg(u) : u \in U\}$, and $\mathrm{mindeg}(V_1, V_2) = \min\{\deg(u, V_2) : u \in V_1\}$ for two sets $V_1, V_2 \subseteq V(G)$. Similar notation is used for the maximum degree, denoted by $\mathrm{maxdeg}(G)$. The neighborhood of a vertex v is denoted by $\mathrm{N}(v)$, and we write $\mathrm{N}(U) = \bigcup_{u \in U} \mathrm{N}(u)$. These symbols have a subscript to emphasize the host graph.

The symbol "$-$" is used for two graph operations: if $U \subseteq V(G)$ is a vertex set, then $G - U$ is the subgraph of G induced by $V(G) \setminus U$. If $H \subseteq G$ is a subgraph of G, then the graph $G-H$ is defined on the vertex set $V(G)$ and corresponds to deletion of edges of H from G.

A family $\mathcal{A}$ of pairwise disjoint subsets of $V(G)$ is an *ℓ-ensemble in G* if $|A| \geqslant \ell$ for each $A \in \mathcal{A}$.

2.2. Regular pairs. We now define regular pairs in the sense of Szemerédi's regularity lemma. Given a graph H and a pair (U,W) of disjoint sets $U,W \subseteq V(H)$, the *density of the pair* (U,W) is defined as

$$\mathrm{d}(U,W) := \frac{e(U,W)}{|U||W|} .$$

Similarly, for a bipartite graph G with color classes U, W, we talk about its *bipartite density* $\mathrm{d}(G) = \frac{e(G)}{|U||W|}$. For a given $\varepsilon > 0$, a pair (U,W) of disjoint sets $U,W \subseteq V(H)$ is called an *ε-regular pair* if $|\mathrm{d}(U,W) - \mathrm{d}(U',W')| < \varepsilon$ for every $U' \subseteq U$, $W' \subseteq W$ with $|U'| \geqslant \varepsilon|U|$, $|W'| \geqslant \varepsilon|W|$. If the pair (U,W) is not ε-regular, then it is called *ε-irregular*. A stronger notion than regularity is that of superregularity, which we recall now. A pair (A,B) is *(ε,γ)-superregular* if it is ε-regular, and both $\mathrm{mindeg}(A,B) \geqslant \gamma|B|$ and $\mathrm{mindeg}(B,A) \geqslant \gamma|A|$. Note that then (A,B) has bipartite density at least γ.

The following facts are well known.

FACT 2.1. *Suppose that (U,W) is an ε-regular pair of density d. Let $U' \subseteq W, W' \subseteq W$ be sets of vertices with $|U'| \geqslant \alpha|U|$, $|W'| \geqslant \alpha|W|$, where $\alpha > \varepsilon$. Then the pair (U',W') is a $2\varepsilon/\alpha$-regular pair of density at least $d-\varepsilon$.*

FACT 2.2. *Suppose that (U,W) is an ε-regular pair of density d. Then all but at most $\varepsilon|U|$ vertices $v \in U$ satisfy $\deg(v,W) \geqslant (d-\varepsilon)|W|$.*

The next lemma asserts that if we have many ε-regular pairs (R,Q_i), then most vertices in R have approximately the total degree into the set $\bigcup_i Q_i$ that we would expect.

LEMMA 2.3. *Let $Q_1,\dots,Q_\ell$ and R be disjoint vertex sets. Suppose further that for each $i \in [\ell]$, the pair (R,Q_i) is ε-regular. Then we have*

(a) $\deg(v,\bigcup_i Q_i) \geqslant \frac{e(R,\bigcup_i Q_i)}{|R|} - \varepsilon\,|\bigcup_i Q_i|$ *for all but at most $\varepsilon|R|$ vertices $v \in R$, and*

(b) $\deg(v,\bigcup_i Q_i) \leqslant \frac{e(R,\bigcup_i Q_i)}{|R|} + \varepsilon\,|\bigcup_i Q_i|$ *for all but at most $\varepsilon|R|$ vertices $v \in R$.*

Proof. We prove (a), and the proof of (b) is similar. Suppose for contradiction that (a) does not hold. Without loss of generality, assume that there is a set $X \subseteq R$, $|X| > \varepsilon|R|$, such that $\frac{e(R,\bigcup Q_i)}{|R|} - \varepsilon|\bigcup Q_i| > \deg(v,\bigcup Q_i)$ for each $v \in X$. By averaging, there is an index $i \in [\ell]$ such that $\frac{|X|}{|R|}e(R,Q_i) - \varepsilon|X||Q_i| > e(X,Q_i)$ or, equivalently, $\mathrm{d}(R,Q_i) - \varepsilon > \mathrm{d}(X,Q_i)$. This contradicts the ε-regularity of the pair (R,Q_i). □

2.3. LKS graphs. We now give some notation specific to our setting. We write $\mathbf{trees}(k)$ for the set of all trees (up to isomorphism) of order k. We write $\mathbf{LKS}(n,k,\alpha)$ for the class of all n-vertex graphs with at least $(\frac{1}{2}+\alpha)n$ vertices of degrees at least $(1+\alpha)k$. With this notation Conjecture 1.1 states that every graph in $\mathbf{LKS}(n,k-1,0)$ contains every tree from $\mathbf{trees}(k)$.

Given a graph G, denote by $\mathbb{S}_{\eta,k}(G)$ the set of those vertices of G that have degree less than $(1+\eta)k$, and by $\mathbb{L}_{\eta,k}(G)$ the set of those vertices of G that have degree at least $(1+\eta)k$.

In [HKP$^+$a] we introduced the class $\mathbf{LKSmin}(n,k,\eta)$ of the graphs that are edge-minimal with respect to membership in $\mathbf{LKS}(n,k,\eta)$. It would be sufficient to prove

Theorem 1.2 for graphs in $\mathbf{LKSmin}(n,k,\eta)$. This class, however, is too rigid with respect to changes that are necessary when applying the sparse decomposition. Therefore, in [HKP$^+$a, section 2.4], we derived a relaxation of the class $\mathbf{LKSmin}(n,k,\eta)$ which we introduce next.

Definition 2.4. *Let* $\mathbf{LKSsmall}(n,k,\eta)$ *be the class of graphs* $G \in \mathbf{LKS}(n,k,\eta)$ *having the following three properties:*

1. *All the neighbors of every vertex* $v \in V(G)$ *with* $\deg(v) > \lceil(1+2\eta)k\rceil$ *have degree at most* $\lceil(1+2\eta)k\rceil$.
2. *All the neighbors of every vertex of* $\mathbb{S}_{\eta,k}(G)$ *have degree exactly* $\lceil(1+\eta)k\rceil$.
3. *We have* $e(G) \leqslant kn$.

2.4. Sparse decomposition. Here we recall some definitions from [HKP$^+$a]: dense spots, avoiding sets, and the key notions of bounded and sparse decomposition. This section is a rather dry list for later reference only, and the reader should consult [HKP$^+$a, section 3] for a more detailed description of these notions. Here, we just recall that the purpose of introducing dense spots, avoiding sets, and nowhere-dense graph is that together with high-degree vertices they form a sparse decomposition of a given graph. The main result of the first paper in the series, [HKP$^+$a, Lemma 3.14], asserts that each graph from $\mathbf{LKS}(n,k,\eta)$ has a sparse decomposition in which almost all edges are of one of the above types. (In fact, the sparse decomposition is not specific to LKS graphs, and indeed in [HKP$^+$a, Lemma 3.15] we provide a corresponding general statement.)

Definition 2.5 ((m,γ)-dense spot, (m,γ)-nowhere-dense). *Suppose that* $m \in \mathbb{N}$ *and* $\gamma > 0$. *An* (m,γ)-dense spot *in a graph* G *is a nonempty bipartite subgraph* $D = (U,W;F)$ *of* G *with* $\mathrm{d}(D) > \gamma$ *and* $\mathrm{mindeg}(D) > m$. *We call a graph* G (m,γ)-nowhere-dense *if it does not contain any* (m,γ)*-dense spot.*

When the parameters m *and* γ *are not relevant, we call* D *simply a* dense spot.

Note that dense spots do not have a specified orientation. That is, we view $(U,W;F)$ and $(W,U;F)$ as the same object.

Definition 2.6 ((m,γ)-dense cover). *Suppose that* $m \in \mathbb{N}$ *and* $\gamma > 0$. *An* (m,γ)-dense cover *of a given graph* G *is a family* $\mathcal{D}$ *of edge-disjoint* (m,γ)*-dense spots such that* $E(G) = \bigcup_{D\in\mathcal{D}} E(D)$.

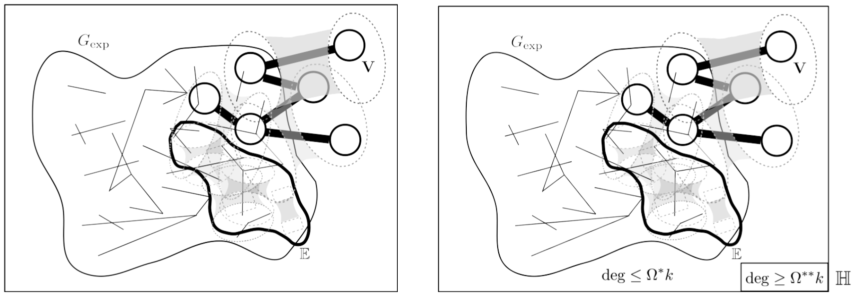
The following two facts are proved in [HKP$^+$a, Facts 3.4 and 3.5].

Fact 2.7. *Let* $(U,W;F)$ *be a* $(\gamma k,\gamma)$*-dense spot in a graph* G *of maximum degree at most* Ωk. *Then* $\max\{|U|,|W|\} \leqslant \frac{\Omega}{\gamma}k$.

Fact 2.8. *Let* H *be a graph of maximum degree at most* Ωk, *let* $v \in V(H)$, *and let* $\mathcal{D}$ *be a family of edge-disjoint* $(\gamma k,\gamma)$*-dense spots. Then fewer than* $\frac{\Omega}{\gamma}$ *dense spots from* $\mathcal{D}$ *contain* v.

We now define the avoiding set. Informally, a set $\mathbb{E}$ of vertices is avoiding if for each set U of size at most Λk (where $\Lambda \gg 1$ is a large constant) and for each vertex $v \in \mathbb{E}$ there is a dense spot containing v and almost disjoint from U. Favorable properties of avoiding sets for embedding trees are shown in [HKP$^+$a, section 3.5].

Definition 2.9 ($(\Lambda,\varepsilon,\gamma,k)$-avoiding set). *Suppose that* $\varepsilon,\gamma > 0$, $\Lambda > 0$, *and* $k \in \mathbb{N}$. *Suppose that* G *is a graph and* $\mathcal{D}$ *is a family of dense spots in* G. *A set* $\mathbb{E} \subseteq \bigcup_{D\in\mathcal{D}} V(D)$ *is* $(\Lambda,\varepsilon,\gamma,k)$-avoiding *with respect to* $\mathcal{D}$ *if for every* $U \subseteq V(G)$ *with* $|U| \leqslant \Lambda k$ *the following holds for all but at most* εk *vertices* $v \in \mathbb{E}$: *There is a dense spot* $D \in \mathcal{D}$ *with* $|U \cap V(D)| \leqslant \gamma^2 k$ *that contains* v.



(a) Bounded decomposition. (b) Sparse decomposition.

FIG. 2. *A simplified illustration of a bounded/sparse decomposition of a graph. The nowhere-dense graph $G_{\exp}$ is shown in black, the dense spots $\mathcal{D}$ in dotted gray (different shades and shapes), the clusters $\mathbf{V}$ and the edges in the cluster graph G_{reg} in thick black, and the avoiding set $\mathbb{E}$ as a thick black region. The difference between the bounded and the sparse decomposition is that no distinction regarding degrees of vertices is made in the former.*

Finally, we can introduce the most important tool in the proof of Theorem 1.2, the *sparse decomposition.* It generalizes the notion of equitable partition from Szemerédi's regularity lemma. The first step towards this end is the notion of bounded decomposition. An illustration is given in Figure 2.

DEFINITION 2.10 ($(k,\Lambda,\gamma,\varepsilon,\nu,\rho)$-bounded decomposition). *Suppose that $\mathcal{V} = \{V_1, V_2, \dots, V_s\}$ is a partition of the vertex set of a graph G. We say that the quintuple $(\mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\exp}, \mathbb{E})$ is a* $(k,\Lambda,\gamma,\varepsilon,\nu,\rho)$-bounded decomposition *of G with respect to $\mathcal{V}$ if the following properties are satisfied:*

1. *$G_{\exp}$ is a $(\gamma k,\gamma)$-nowhere-dense subgraph of G with* $\mathrm{mindeg}(G_{\exp}) > \rho k$.
2. *$\mathbf{V}$ consists of disjoint subsets of $V(G)$.*
3. *G_{reg} is a subgraph of $G - G_{\exp}$ on the vertex set $\bigcup \mathbf{V}$. For each edge $xy \in E(G_{\mathrm{reg}})$ there are distinct $C_x \ni x$ and $C_y \ni y$ from $\mathbf{V}$, and $G[C_x, C_y] = G_{\mathrm{reg}}[C_x, C_y]$. Furthermore, $G[C_x, C_y]$ forms an ε-regular pair of density at least γ^2.*
4. *We have $\nu k \leqslant |C| = |C'| \leqslant \varepsilon k$ for all $C, C' \in \mathbf{V}$.*
5. *$\mathcal{D}$ is a family of edge-disjoint $(\gamma k,\gamma)$-dense spots in $G - G_{\exp}$. For each $D = (U, W; F) \in \mathcal{D}$ all the edges of $G[U, W]$ are covered by $\mathcal{D}$ (but not necessarily by D).*
6. *If G_{reg} contains at least one edge between $C_1 \in \mathbf{V}$ and $C_2 \in \mathbf{V}$ then there exists a dense spot $D = (U, W; F) \in \mathcal{D}$ such that $C_1 \subseteq U$ and $C_2 \subseteq W$.*
7. *For each $C \in \mathbf{V}$ there is $V \in \mathcal{V}$ so that either $C \subseteq V \cap V(G_{\exp})$ or $C \subseteq V \setminus V(G_{\exp})$. For each $C \in \mathbf{V}$ and each $D = (U, W; F) \in \mathcal{D}$ we have that either C is disjoint from D or contained in D.*
8. *$\mathbb{E}$ is a $(\Lambda, \varepsilon, \gamma, k)$-avoiding subset of $V(G) \setminus \bigcup \mathbf{V}$ with respect to dense spots $\mathcal{D}$.*

We say that the bounded decomposition $(\mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\exp}, \mathbb{E})$ respects the avoiding threshold *b if for each $C \in \mathbf{V}$ we either have* $\mathrm{maxdeg}_G(C, \mathbb{E}) \leqslant b$, *or* $\mathrm{mindeg}_G(C, \mathbb{E}) > b$.

The members of $\mathbf{V}$ are called *clusters.* Define the *cluster graph* $\mathbf{G}_{\mathrm{reg}}$ as the graph on the vertex set $\mathbf{V}$ that has an edge C_1C_2 for each pair (C_1, C_2) which has density at least γ^2 in the graph G_{reg}.

We can now introduce the notion of sparse decomposition in which we enhance a bounded decomposition by distinguishing between vertices of huge and moderate degree.

DEFINITION 2.11 ($(k,\Omega^{**},\Omega^*,\Lambda,\gamma,\varepsilon,\nu,\rho)$-sparse decomposition). *Suppose that $k\in\mathbb{N}$, $\varepsilon,\gamma,\nu,\rho>0$, and $\Lambda,\Omega^*,\Omega^{**}>0$. Let $\mathcal{V}=\{V_1,V_2,\dots,V_s\}$ be a partition of the vertex set of a graph G. We say that $\nabla=(\mathbb{H},\mathbf{V},\mathcal{D},G_{\mathrm{reg}},G_{\mathrm{exp}},\mathbb{E})$ is a* $(k,\Omega^{**},\Omega^*,\Lambda,\gamma,\varepsilon,\nu,\rho)$-sparse decomposition *of G with respect to $V_1,V_2,\dots,V_s$ if the following hold:*

1. $\mathbb{H}\subseteq V(G)$, $\mathrm{mindeg}_G(\mathbb{H})\geqslant\Omega^{**}k$, $\mathrm{maxdeg}_K(V(G)\setminus\mathbb{H})\leqslant\Omega^*k$, *where K is spanned by the edges of $\bigcup\mathcal{D}$, G_{exp}, and edges to with $\mathbb{H}$,*
2. $(\mathbf{V},\mathcal{D},G_{\mathrm{reg}},G_{\mathrm{exp}},\mathbb{E})$ *is a $(k,\Lambda,\gamma,\varepsilon,\nu,\rho)$-bounded decomposition of $G-\mathbb{H}$ with respect to $V_1\setminus\mathbb{H},V_2\setminus\mathbb{H},\dots,V_s\setminus\mathbb{H}$.*

If the parameters do not matter, we call ∇ simply a *sparse decomposition*, and similarly we speak about a *bounded decomposition*.

DEFINITION 2.12 (captured edges, graphs G_∇ and $G_{\mathcal{D}}$). *In the situation of Definition* 2.11*, we define the graph $G_{\mathcal{D}}$ as the graph induced by the dense spots, i.e., $V(G_{\mathcal{D}})=\bigcup_{D\in\mathcal{D}}V(D)$, $E(G_{\mathcal{D}})=\bigcup_{D\in\mathcal{D}}E(D)$.*

We refer to the edges in $E(G_{\mathrm{reg}})\cup E(G_{\mathrm{exp}})\cup E_G(\mathbb{H},V(G))\cup E_{G_{\mathcal{D}}}(\mathbb{E},\mathbb{E}\cup\bigcup\mathbf{V})$ as captured *by the sparse decomposition. We write G_∇ for the subgraph of G on the same vertex set which consists of the captured edges.*

Likewise, the captured edges *of a bounded decomposition $(\mathbf{V},\mathcal{D},G_{\mathrm{reg}},G_{\mathrm{exp}},\mathbb{E})$ of a graph G are those in $E(G_{\mathrm{reg}})\cup E(G_{\mathrm{exp}})\cup E_{G_{\mathcal{D}}}(\mathbb{E},\mathbb{E}\cup\bigcup\mathbf{V})$.*

2.5. Regularized matchings. We recall the notion of a regularized matching, introduced in [HKP+b].[1]

DEFINITION 2.13 ((ε,d,ℓ)-regularized matching). *Suppose that $\ell\in\mathbb{N}$ and $d,\varepsilon>0$. A collection $\mathcal{N}$ of pairs (A,B) with $A,B\subseteq V(H)$ is called an* (ε,d,ℓ)-regularized matching *of a graph H if*

- (i) $|A|=|B|\geqslant\ell$ *for each* $(A,B)\in\mathcal{N}$,
- (ii) (A,B) *induces in H an ε-regular pair of density at least d for each* $(A,B)\in\mathcal{N}$, *and*
- (iii) *the sets* $\{A\}_{(A,B)\in\mathcal{N}}$ *and* $\{B\}_{(A,B)\in\mathcal{N}}$ *are pairwise disjoint.*

Sometimes, when the parameters do not matter, we simply write regularized matching.

Suppose that $\mathcal{N}$ is a regularized matching, and $(A,B)\in\mathcal{N}$. Then we call A a *partner* of B and B a partner of A (in $\mathcal{N}$).

We shall make use of some auxiliary results from [HKP+b]. To this end, we need a definition.

DEFINITION 2.14 (see [HKP+b, Definition 3.7]). *We define $\mathcal{G}(n,k,\Omega,\rho,\nu,\tau)$ to be the class of all tuples $(G,\mathcal{D},H,\mathcal{A})$ with the following properties:*

- (i) *G is a graph of order n with* $\mathrm{maxdeg}(G)\leqslant\Omega k$.
- (ii) *H is a bipartite subgraph of G with color classes A_H and B_H and with $e(H)\geqslant\tau kn$.*
- (iii) *$\mathcal{D}$ is a $(\rho k,\rho)$-dense cover of G.*
- (iv) *$\mathcal{A}$ is a (νk)-ensemble in G, and $A_H\subseteq\bigcup\mathcal{A}$.*
- (v) *$A\cap U\in\{\emptyset,A\}$ for each $A\in\mathcal{A}$ and for each $D=(U,W;F)\in\mathcal{D}$.*

[1] In older versions of [HKP+b, HKP+d] (the arXiv preprints) and in the published version of [HPS+15], we used the name "semiregular matchings."

LEMMA 2.15 (see [HKP+b, Lemma 4.4]). *For every* $\bar{\Omega} \in \mathbb{N}$ *and* $\bar{\rho}, \bar{\varepsilon}, \bar{\tau} \in (0,1)$ *there exists an* $\bar{\alpha} > 0$ *such that for every* $\bar{\nu} \in (0,1)$ *there is a number* $\bar{k}_0 \in \mathbb{N}$ *such that the following holds for every* $k > \bar{k}_0$.

For each $(\bar{G}, \bar{\mathcal{D}}, \bar{H}, \bar{\mathcal{A}}) \in \mathcal{G}(n, k, \bar{\Omega}, \bar{\rho}, \bar{\nu}, \bar{\tau})$ *there exists an* $(\bar{\varepsilon}, \frac{\bar{\tau}\bar{\rho}}{8\bar{\Omega}}, \bar{\alpha}\bar{\nu}k)$*-regularized matching* $\bar{\mathcal{M}}$ *of* $\bar{G}$ *such that*

(1) *for each* $(X, Y) \in \bar{\mathcal{M}}$ *there are* $A \in \bar{\mathcal{A}}$ *and* $D = (U, W; F) \in \bar{\mathcal{D}}$ *such that* $X \subseteq U \cap A \cap A_H$ *and* $Y \subseteq W \cap B_H$, *and*

(2) $|V(\bar{\mathcal{M}})| \geqslant \frac{\bar{\tau}}{2\bar{\Omega}} n$.

2.6. Cutting trees. We outline the way we process any k-vertex tree T in our proof of Theorem 1.2. This is done in detail in [HKP+d, section 3]. The purpose of the informal description below is only to serve as a reference when we motivate the configurations in section 4.1.

Given T, we introduce a constant number (i.e., independent of k) of *cut-vertices* $W \subseteq V(T)$. We can do so in such a way that the following properties are satisfied:[2]

- The set W is partitioned into sets $W_A \dot\cup W_B$ such that the distance between each vertex of W_A and each vertex of W_B is odd.
- The trees of $T - W$, which are called *shrubs*, are all small, i.e., of order $O(\frac{k}{|W|})$. Each shrub neighbors either one vertex of W (in which case it is called an *end shrub*) or two vertices of W (in which case it is called an *internal shrub*).
- The two neighbors in W of each internal shrub are from W_A.
- The components of $T[W]$ are referred to as *hubs*.
- The shrubs that neighbor a vertex (or two vertices) of W_A are denoted $\mathcal{S}_A$. The shrubs that neighbor a vertex of W_B are denoted $\mathcal{S}_B$.

We call the quadruple $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ a *fine partition* of T.

3. Shadows, random splitting, and common settings. In this section we will prove some preliminaries needed for the main results of this paper, presented in section 4. The present section is organized as follows. In section 3.1, we introduce an auxiliary notion of shadows and prove some simple properties. Section 3.2 introduces randomized splitting of the vertex set of an input graph. In section 3.3, we introduce building blocks for the finer structure, which we will obtain in section 4.

3.1. Shadows. We will find it convenient to work with the notion of a shadow. To motivate this notion, we recall the greedy embedding strategy. Suppose that T is a tree of order k and G is a graph with minimum degree at least $k-1$. We can then root T at an arbitrary vertex. Then, we embed that vertex in G. Now, at each step, we have a partial embedding of T in G. We pick one vertex of T that is already embedded but whose children are still unembedded, and we embed those in T. The minimum-degree condition tells us that we can always accommodate these children.

The greedy embedding strategy clearly fails in the setting of Theorem 1.2. So, we need to enhance the strategy by not embedding the vertices of $T_{\mathrm{T1.2}}$ in some part U (which is not suitable for continuing the embedding) of $G_{\mathrm{T1.2}}$. This forces us to look ahead: When embedding a vertex v of $T_{\mathrm{T1.2}}$, we want to avoid not only U, but also vertices that send many edges to U, since we want to avoid U also with the children of v. The notion of shadow formalizes this.

[2]Here, we list only properties that are relevant for the description later. See [HKP+d, Definition 3.3 and Lemma 3.5] for details.

Given a graph H, a set $U \subseteq V(H)$, and a number ℓ, we define inductively

$$\mathrm{shadow}^{(0)}_H(U,\ell) := U \;,$$
$$\mathrm{shadow}^{(i)}_H(U,\ell) := \left\{v \in V(H) \,:\, \deg_H(v, \mathrm{shadow}^{(i-1)}_H(U,\ell)) > \ell\right\} \text{ for } i \geqslant 1 \;.$$

We abbreviate $\mathrm{shadow}^{(1)}_H(U,\ell)$ as $\mathrm{shadow}_H(U,\ell)$. Further, the graph H is omitted from the subscript if it is clear from the context. Note that the shadow of a set U might intersect U.

Below, we state two facts which bound the size of a shadow of a given set. Fact 3.1 gives a bound for general graphs of bounded maximum degree, and Fact 3.2 gives a stronger bound for nowhere-dense graphs.

FACT 3.1. *Suppose H is a graph with* $\mathrm{maxdeg}(H) \leqslant \Omega k$. *Then for each $\alpha > 0$, $i \in \{0,1,\ldots\}$, and each set $U \subseteq V(H)$, we have*

$$|\mathrm{shadow}^{(i)}(U,\alpha k)| \leqslant \left(\frac{\Omega}{\alpha}\right)^i |U| \;.$$

Proof. Proceeding by induction on i, it suffices to show that $|\mathrm{shadow}^{(1)}(U,\alpha k)| \leqslant \Omega|U|/\alpha$. To this end, observe that U sends out at most $\Omega k|U|$ edges, while each vertex of $\mathrm{shadow}(U,\alpha k)$ receives at least αk edges from U. ∎

FACT 3.2. *Let $\alpha,\gamma,Q > 0$ be three numbers such that $1 \leqslant Q \leqslant \frac{\alpha}{16\gamma}$. Suppose that H is a $(\gamma k,\gamma)$-nowhere-dense graph, and let $U \subseteq V(H)$ with $|U| \leqslant Qk$. Then we have*

$$|\mathrm{shadow}(U,\alpha k)| \leqslant \frac{16Q^2\gamma}{\alpha}k \;.$$

Proof. Suppose the contrary, and let $W \subseteq \mathrm{shadow}(U,\alpha k)$ be of size $|W| = \frac{16Q^2\gamma}{\alpha}k \leqslant Qk$. Then $e_H(U \cup W) \geqslant \frac{1}{2}\sum_{v\in W}\deg_H(v,U) \geqslant 8\gamma Q^2k^2$. Thus $H[U\cup W]$ has average degree at least

$$\frac{2e_H(U\cup W)}{|U|+|W|} \geqslant 8\gamma Qk \;,$$

and therefore, by a well-known fact, contains a subgraph H' of minimum degree at least $4\gamma Qk$. Taking a maximal cut (A,B) in H', it is easy to see that $H'[A,B]$ has minimum degree at least $2\gamma Qk \geqslant \gamma k$. Further, $H'[A,B]$ has density at least $\frac{|A|\cdot 2\gamma Qk}{|A||B|} \geqslant \gamma$, contradicting that H is $(\gamma k,\gamma)$-nowhere-dense. ∎

3.2. Random splitting. Suppose a graph G (together with its bounded decomposition) is given. In this section we split its vertex set into several classes, the sizes of which have given ratios. It is important that most vertices will have their degrees split obeying approximately these ratios. The corresponding statement is given in Lemma 3.3. It will be used to split the vertices of the host graph $G = G_{\mathrm{T1.2}}$ according to which part of the tree $T = T_{\mathrm{T1.2}} \in \mathbf{trees}(k)$ they will host. More precisely, suppose that $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ is a fine partition of T. Let t_{int} and t_{end} be the total sizes of the internal and end shrubs, respectively. We then want to partition $V(G)$ into three sets $\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2$ in the ratio (approximately)

$$(|W_A| + |W_B|) \,:\, t_{\mathrm{int}} \,:\, t_{\mathrm{end}}$$

so that degrees of the vertices of $V(G)$ are split proportionally. This will allow us to embed the vertices of $W_A \cup W_B$ into $\mathbb{A}_0$, the internal shrubs into $\mathbb{A}_1$, and end shrubs into $\mathbb{A}_2$. Actually, since our embedding procedure is more complex, we require not only that the degrees be split proportionally, but also that the objects from the bounded decomposition be partitioned proportionally. In [HKP$^+$d] it will become clearer why such a random splitting needs to be used.

Lemma 3.3 is formulated in an abstract setting, without any reference to the tree T and with a general number of classes in the partition.

LEMMA 3.3. *For each $p \in \mathbb{N}$ and $a > 0$ there exists $k_0 > 0$ such that for each $k > k_0$ we have the following.*

Suppose that G is a graph of order $n \geqslant k_0$ and $\mathrm{maxdeg}(G) \leqslant \Omega^* k$ with its $(k, \Lambda, \gamma, \varepsilon, k^{-0.05}, \rho)$-bounded decomposition $(\mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathbb{E})$. As usual, we write G_∇ for the subgraph captured by $(\mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathbb{E})$, and $G_\mathcal{D}$ for the spanning subgraph of G consisting of the edges in $\mathcal{D}$. Let $\mathcal{M}$ be an $(\varepsilon, d, k^{0.95})$-regularized matching in G, and let $\mathbb{B}_1, \ldots, \mathbb{B}_p$ be subsets of $V(G)$. Suppose that $\Omega^* \geqslant 1$ and $\Omega^*/\gamma < k^{0.1}$.

Suppose that $\mathfrak{q}_1, \ldots, \mathfrak{q}_p \in \{0\} \cup [a, 1]$ are reals with $\sum \mathfrak{q}_i \leqslant 1$. Then there exist a partition $\mathbb{A}_1 \cup \cdots \cup \mathbb{A}_p = V(G)$ and sets $\bar{V} \subseteq V(G)$, $\bar{\mathcal{V}} \subseteq \mathcal{V}(\mathcal{M})$, $\bar{\mathbf{V}} \subseteq \mathbf{V}$ with the following properties:

(1) $|\bar{V}| \leqslant \exp(-k^{0.1})n$, $|\bigcup \bar{\mathcal{V}}| \leqslant \exp(-k^{0.1})n$, $|\bigcup \bar{\mathbf{V}}| < \exp(-k^{0.1})n$.
(2) *For each $i \in [p]$ and each $C \in \mathbf{V} \setminus \bar{\mathbf{V}}$ we have $|C \cap \mathbb{A}_i| \geqslant \mathfrak{q}_i |\mathbb{A}_i| - k^{0.9}$.*
(3) *For each $i \in [p]$ and each $C \in \mathcal{V}(\mathcal{M}) \setminus \bar{\mathcal{V}}$ we have $|C \cap \mathbb{A}_i| \geqslant \mathfrak{q}_i |\mathbb{A}_i| - k^{0.9}$.*
(4) *For each $i \in [p]$, $D = (U, W; F) \in \mathcal{D}$ and $\mathrm{mindeg}_D(U \setminus \bar{V}, W \cap \mathbb{A}_i) \geqslant \mathfrak{q}_i \gamma k - k^{0.9}$.*
(5) *For each $i, j \in [p]$ we have $|\mathbb{A}_i \cap \mathbb{B}_j| \geqslant \mathfrak{q}_i |\mathbb{B}_j| - n^{0.9}$.*
(6) *For each $i \in [p]$, each $J \subseteq [p]$, and each $v \in V(G) \setminus \bar{V}$ we have*

$$\deg_H(v, \mathbb{A}_i \cap \mathbb{B}_J) \geqslant \mathfrak{q}_i \deg_H(v, \mathbb{B}_J) - 2^{-p} k^{0.9}$$

for each graph $H \in \{G, G_\nabla, G_{\mathrm{exp}}, G_\mathcal{D}, G_\nabla \cup G_\mathcal{D}\}$, where $\mathbb{B}_J := (\bigcap_{j \in J} \mathbb{B}_j) \setminus (\bigcup_{j \in [p] \setminus J} \mathbb{B}_j)$.

(7) *For each $i, i', j, j' \in [p]$ $(j \neq j')$, we have*

$$e_H(\mathbb{A}_i \cap \mathbb{B}_j, \mathbb{A}_{i'} \cap \mathbb{B}_{j'}) \geqslant \mathfrak{q}_i \mathfrak{q}_{i'} e_H(\mathbb{B}_j, \mathbb{B}_{j'}) - k^{0.6} n^{0.6} ,$$
$$e_H(\mathbb{A}_i \cap \mathbb{B}_j, \mathbb{A}_{i'} \cap \mathbb{B}_j) \geqslant \mathfrak{q}_i \mathfrak{q}_{i'} e(H[\mathbb{B}_j]) - k^{0.6} n^{0.6} \qquad \text{if } i \neq i' ,$$
$$e(H[\mathbb{A}_i \cap \mathbb{B}_j]) \geqslant \mathfrak{q}_i^2 e(H[\mathbb{B}_j]) - k^{0.6} n^{0.6}$$

for each graph $H \in \{G, G_\nabla, G_{\mathrm{exp}}, G_\mathcal{D}, G_\nabla \cup G_\mathcal{D}\}$.

(8) *For each $i \in [p]$ if $\mathfrak{q}_i = 0$, then $\mathbb{A}_i = \emptyset$.*

Proof. We can assume that $\sum \mathfrak{q}_i = 1$ since all bounds in (2)–(7) are lower bounds. Assume that k is large enough. We assign each vertex $v \in V(G)$ to one of the sets $\mathbb{A}_1, \ldots, \mathbb{A}_p$ at random with respective probabilities $\mathfrak{q}_1, \ldots, \mathfrak{q}_p$. Let $\bar{V}_1$ and $\bar{V}_2$ be the vertices which do not satisfy (4) and (6), respectively. Let $\bar{\mathcal{V}}$ be the sets of $\mathcal{V}(\mathcal{M})$ which do not satisfy (3), and let $\bar{\mathbf{V}}$ be the clusters of $\mathbf{V}$ which do not satisfy (2). Setting $\bar{V} := \bar{V}_1 \cup \bar{V}_2$, we need to show that (1), (5), and (7) are fulfilled simultaneously with positive probability. Using the union bound, it suffices to show that each of the properties (1), (5), and (7) is violated with probability at most 0.2. The probability of each of these three properties can be controlled in a straightforward way by the Chernoff bound. We only give such a bound (with error probability at most 0.1) on the size of the set $\bar{V}_1$ (appearing in (1)), which is the most difficult one to control.

For $i \in [p]$, let $\bar{V}_{1,i}$ be the set of vertices v for which there exists $D = (U, W; F) \in \mathcal{D}$, with $U \ni v$, such that $\deg_D(v, W \cap \mathbb{A}_i) < \mathfrak{q}_i \gamma k - k^{0.9}$. We aim to show that for each $i \in [p]$ the probability that $|\bar{V}_{1,i}| > \exp(-k^{0.2})n$ is at most $\frac{1}{10p}$. Indeed, summing such an error bound together with similar bounds for other properties will allow us to conclude with the statement. This will in turn follow from the Markov inequality, provided that we show that

$$\mathbf{E}[|\bar{V}_{1,i}|] \leqslant \frac{1}{10p} \cdot \exp(-k^{0.2})n \ . \tag{3.1}$$

Indeed, let us consider an arbitrary vertex $v \in V(G)$. By Fact 2.8, v is contained in at most Ω^*/γ dense spots of $\mathcal{D}$. For a fixed dense spot $D = (U, W; F) \in \mathcal{D}$ with $v \in U$ let us bound the probability of the event $\mathcal{E}_{v,i,D}$ that $\deg_D(v, W \cap \mathbb{A}_i) < \mathfrak{q}_i \gamma k - k^{0.9}$. To this end, fix a set $N \subseteq W \cap \mathrm{N}_D(v)$ of size exactly γk before the random assignment is performed. Now, elements of $V(G)$ are distributed randomly into the sets $\mathbb{A}_1, \ldots, \mathbb{A}_p$. In particular, the number $|\mathbb{A}_i \cap N|$ has binomial distribution with parameters γk and $\mathfrak{q}_i$. Using the Chernoff bound, we get

$$\mathbf{P}[\mathcal{E}_{v,i,D}] \leqslant \mathbf{P}\left[|\mathbb{A}_i \cap N| < \mathfrak{q}_i \gamma k - k^{0.9}\right] \leqslant \exp(-k^{0.3}) \ .$$

Thus, it follows by summing the tail over at most $\Omega^*/\gamma \leqslant k^{0.1}$ dense spots containing v that

$$\mathbf{P}[v \in \bar{V}_{1,i}] \leqslant k^{0.1} \cdot \exp(-k^{0.3}) \ . \tag{3.2}$$

Now, (3.1) follows by linearity of expectation. □

Lemma 3.3 is utilized for the purpose of our proof of Theorem 1.2 using the notion of proportional partition introduced in Definition 3.7 below.

3.3. Common settings. Throughout section 3 we shall be working with the setting that comes from [HKP+b, Lemma 5.4]. To keep statements of the subsequent lemmas reasonably short, we introduce a common setting.

Suppose that G is a graph with a $(k, \Omega^{**}, \Omega^*, \Lambda, \gamma, \varepsilon, \nu, \rho)$-sparse decomposition

$$\nabla = (\mathbb{H}, \mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathbb{E})$$

with respect to $(\mathbb{L}_{\eta,k}(G), \mathbb{S}_{\eta,k}(G))$. Suppose further that $\mathcal{M}_A, \mathcal{M}_B$ are $(\varepsilon', d, \gamma k)$-regularized matchings in $G_{\mathcal{D}}$. The triple $(\mathbb{XA}, \mathbb{XB}, \mathbb{XC}) = (\mathbb{XA}, \mathbb{XB}, \mathbb{XC})(\eta, \nabla, \mathcal{M}_A, \mathcal{M}_B)$ is then defined by setting

$$\begin{aligned}
\mathbb{XA} &:= \mathbb{L}_{\eta,k}(G) \setminus V(\mathcal{M}_B) \ , \\
\mathbb{XB} &:= \left\{ v \in V(\mathcal{M}_B) \cap \mathbb{L}_{\eta,k}(G) \ : \ \widehat{\deg}(v) < (1+\eta)\frac{k}{2} \right\} \ , \\
\mathbb{XC} &:= \mathbb{L}_{\eta,k}(G) \setminus (\mathbb{XA} \cup \mathbb{XB}) \ ,
\end{aligned}$$

where $\widehat{\deg}(v)$ on the second line is defined by

$$\widehat{\deg}(v) := \deg_G\left(v, \mathbb{S}_{\eta,k}(G) \setminus \left(V(G_{\mathrm{exp}}) \cup \mathbb{E} \cup V(\mathcal{M}_A \cup \mathcal{M}_B)\right)\right) \ . \tag{3.3}$$

Remark 3.4. The sets $\mathbb{XA}, \mathbb{XB}, \mathbb{XC}$ were defined in [HKP$^+$b, Definition 5.3]. Of course, in applications, the matchings $\mathcal{M}_A$ and $\mathcal{M}_B$ will be guaranteed to have some favorable properties. These properties are formulated in [HKP$^+$b, Lemma 5.4] and are listed in (1)–(8) of Setting 3.5 below. It was argued in [HKP$^+$b, section 5.1] why the set $\mathbb{XA}$ has excellent properties for accommodating cut-vertices of $T_{\mathrm{T1.2}}$, and the set $\mathbb{XB}$ has "half as excellent properties" for accommodating cut-vertices. In particular, the formula defining $\mathbb{XB}$ suggests that we cannot make use of the set $\mathbb{S}_{\eta,k}(G) \setminus (V(G_{\mathrm{exp}}) \cup \mathbb{E} \cup V(\mathcal{M}_A \cup \mathcal{M}_B))$ for the purpose of embedding shrubs neighboring the cut-vertices embedded into $\mathbb{XB}$.

With this notation, we can introduce the common setting, Setting 3.5. This setting serves as an interface between what has been done in [HKP$^+$a, HKP$^+$b] and what will be needed in [HKP$^+$d]. Thus, where possible, we interlace the (highly technical) definitions of Setting 3.5 with some motivation and references.

SETTING 3.5. *We assume that the constants* $\Lambda, \Omega^*, \Omega^{**}, k_0$ *and* $\widehat{\alpha}, \gamma, \varepsilon, \varepsilon', \eta, \pi, \rho, \tau, d$ *satisfy*

$$\frac{1}{2} \geqslant \eta \gg \frac{1}{\Omega^*} \gg \frac{1}{\Omega^{**}} \gg \rho \gg \gamma \gg d \geqslant \frac{1}{\Lambda} \geqslant \varepsilon \geqslant \pi \geqslant \widehat{\alpha} \geqslant \varepsilon' \geqslant \nu \gg \tau \gg \frac{1}{k_0} > 0\ , \tag{3.4}$$

and that $k \geqslant k_0$. *Here, by writing* $c > a_1 \gg a_2 \gg \cdots \gg a_\ell > 0$ *we mean that there exist suitable nondecreasing functions* $f_i : (0,c)^i \to (0,c)$ $(i = 1, \ldots, \ell-1)$ *such that for each* $i \in [\ell-1]$ *we have* $a_{i+1} < f_i(a_1, \ldots, a_i)$. *A suitable choice of these functions in* (3.4) *is determined by the properties we require in* [HKP$^+$d].

Suppose that $G \in \mathbf{LKSsmall}(n, k, \eta)$ *is given with its* $(k, \Omega^{**}, \Omega^*, \Lambda, \gamma, \varepsilon', \nu, \rho)$-*sparse decomposition*

$$\nabla = (\mathbb{H}, \mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathbb{E})\ ,$$

with respect to the partition $\{\mathbb{S}_{\eta,k}(G), \mathbb{L}_{\eta,k}(G)\}$ *and with respect to the avoiding threshold* $\frac{\rho k}{100\Omega^*}$. *We write*

$$V_{\leadsto\mathbb{E}} := \mathrm{shadow}_{G_\nabla - \mathbb{H}}\left(\mathbb{E}, \frac{\rho k}{100\Omega^*}\right) \quad \textit{and} \quad \mathbf{V}_{\leadsto\mathbb{E}} := \{C \in \mathbf{V} \,:\, C \subseteq V_{\leadsto\mathbb{E}}\}\ . \tag{3.5}$$

The graph $\mathbf{G}_{\mathrm{reg}}$ *is the corresponding cluster graph. Let* $\mathfrak{c}$ *be the size of an arbitrary cluster*[3] *in* $\mathbf{V}$. *Let* G_∇ *be the spanning subgraph of* G *formed by the edges captured by* ∇. *There are two* $(\varepsilon, d, \pi\mathfrak{c})$-*regularized matchings* $\mathcal{M}_A$ *and* $\mathcal{M}_B$ *in* $G_{\mathcal{D}}$, *with the following properties (we abbreviate* $\mathbb{XA} := \mathbb{XA}(\eta, \nabla, \mathcal{M}_A, \mathcal{M}_B)$, $\mathbb{XB} := \mathbb{XB}(\eta, \nabla, \mathcal{M}_A, \mathcal{M}_B)$, *and* $\mathbb{XC} := \mathbb{XC}(\eta, \nabla, \mathcal{M}_A, \mathcal{M}_B)$*):*[4]

(1) $V(\mathcal{M}_A) \cap V(\mathcal{M}_B) = \emptyset$.

(2) $V_1(\mathcal{M}_B) \subseteq S^0$, *where*

$$S^0 := \mathbb{S}_{\eta,k}(G) \setminus (V(G_{\mathrm{exp}}) \cup \mathbb{E})\ . \tag{3.6}$$

(3) *For each* $(X, Y) \in \mathcal{M}_A \cup \mathcal{M}_B$, *there is a dense spot* $(U, W; F) \in \mathcal{D}$ *with* $X \subseteq U$ *and* $Y \subseteq W$, *and further, either* $X \subseteq \mathbb{S}_{\eta,k}(G)$ *or* $X \subseteq \mathbb{L}_{\eta,k}(G)$, *and either* $Y \subseteq \mathbb{S}_{\eta,k}(G)$ *or* $Y \subseteq \mathbb{L}_{\eta,k}(G)$.

[3]The number $\mathfrak{c}$ is not defined when $\mathbf{V} = \emptyset$. However, in that case $\mathfrak{c}$ is never actually used.

[4]Let us note that properties (1)–(8) come from [HKP$^+$b, Lemma 5.4], and properties (9) and (10) come from [HKP$^+$a, Lemma 3.14].

(4) *For each* $X_1 \in \mathcal{V}_1(\mathcal{M}_A \cup \mathcal{M}_B)$ *there exists a cluster* $C_1 \in \mathbf{V}$ *such that* $X_1 \subseteq C_1$*, and for each* $X_2 \in \mathcal{V}_2(\mathcal{M}_A \cup \mathcal{M}_B)$ *there exists* $C_2 \in \mathbf{V} \cup \{\mathbb{L}_{\eta,k}(G) \cap \mathbb{E}\}$ *such that* $X_2 \subseteq C_2$*.*

(5) *Each pair of the regularized matching* $\mathcal{M}_{\mathrm{good}} := \{(X_1, X_2) \in \mathcal{M}_A : X_1 \cup X_2 \subseteq \mathbb{XA}\}$ *corresponds to an edge in* $\mathbf{G}_{\mathrm{reg}}$*.*

(6) $e_{G_\nabla}\big(\mathbb{XA}, S^0 \setminus V(\mathcal{M}_A)\big) \leqslant \gamma kn$*.*

(7) $e_{G_{\mathrm{reg}}}(V(G) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)) \leqslant \gamma^2 kn$*.*

(8) *For the regularized matching* $\mathcal{N}_{\mathbb{E}} := \{(X, Y) \in \mathcal{M}_A \cup \mathcal{M}_B : (X \cup Y) \cap \mathbb{E} \neq \emptyset\}$ *we have* $e_{G_{\mathrm{reg}}}\big(V(G) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B), V(\mathcal{N}_{\mathbb{E}})\big) \leqslant \gamma^2 kn$*.*

(9) $|E(G) \setminus E(G_\nabla)| \leqslant 2\rho kn$*.*

(10) $|E(G_{\mathcal{D}}) \setminus (E(G_{\mathrm{reg}}) \cup E_G[\mathbb{E}, \mathbb{E} \cup \bigcup \mathbf{V}])| \leqslant \frac{5}{4}\gamma kn$*.*

We now define several additional vertex sets. The first, the set V_+*, is just the complement of the set used in* (3.3)*,*

$$V_+ := V(G) \setminus (S^0 \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)) \tag{3.7}$$
$$= \mathbb{L}_{\eta,k}(G) \cup V(G_{\mathrm{exp}}) \cup \mathbb{E} \cup V(\mathcal{M}_A \cup \mathcal{M}_B)\ . \tag{3.8}$$

The set $L_\#$ *defined below is the set of "bad vertices of* $\mathbb{L}_{\eta,k}(G)$*," that is, the set of those vertices which have many uncaptured neighbors in the sparse decomposition. If we think of the set* V_+ *as candidate vertices for embedding certain shrubs (cf. Remark* 3.4*), then we'd better discard vertices with a big uncaptured degree from that set. This leads us to the definition of the set* V_{good}*. Since the set* $\mathbb{H}$ *is treated separately, it is also deleted from* V_{good}*.*

$$L_\# := \mathbb{L}_{\eta,k}(G) \setminus \mathbb{L}_{\frac{9}{10}\eta,k}(G_\nabla)\ , \tag{3.9}$$
$$V_{\mathrm{good}} := V_+ \setminus (\mathbb{H} \cup L_\#)\ . \tag{3.10}$$

We can now define sets $\mathbb{YA}$ *and* $\mathbb{YB}$*, which should be regarded as cleaned versions of the sets* $\mathbb{XA}$ *and* $\mathbb{XB}$*. Here, by a* cleaning *we mean the process of getting rid of certain atypical vertices. Indeed, Lemma* 3.10 *below asserts that* $\mathbb{YA}$ *approximately equals* $\mathbb{XA}$*, and* $\mathbb{YB}$ *approximately equals* $\mathbb{XB}$*. Set*

$$\mathbb{YA} := \mathrm{shadow}_{G_\nabla}\left(V_+ \setminus L_\#, \left(1 + \frac{\eta}{10}\right) k\right) \setminus \mathrm{shadow}_{G - G_\nabla}\left(V(G), \frac{\eta}{100} k\right), \tag{3.11}$$
$$\mathbb{YB} := \mathrm{shadow}_{G_\nabla}\left(V_+ \setminus L_\#, \left(1 + \frac{\eta}{10}\right) \frac{k}{2}\right) \setminus \mathrm{shadow}_{G - G_\nabla}\left(V(G), \frac{\eta}{100} k\right). \tag{3.12}$$

When the set $\mathbb{H}$ *is negligible, the configuration we obtain does not involve* $\mathbb{H}$ *at all. In other words,* $\mathbb{H}$ *is not used for embedding. Thus, we use the concept of shadows in the way described at the beginning of section* 3.1 *to avoid* $\mathbb{H}$ *and define* $V_{\rightsquigarrow\mathbb{H}}$ *as follows:*

$$V_{\rightsquigarrow\mathbb{H}} := (\mathbb{XA} \cup \mathbb{XB}) \cap \mathrm{shadow}_G\left(\mathbb{H}, \tfrac{\eta}{100} k\right)\ . \tag{3.13}$$

Next, we define "bad sets" $\mathbb{J}_{\mathbb{E}}$*,* $\mathbb{J}_1$*,* $\mathbb{J}$*,* $\mathbb{J}_2$*, and* $\mathbb{J}_3$*, again using shadows:*

$$\begin{aligned}
\mathbb{J}_{\mathbb{E}} &:= \mathrm{shadow}_{G_{\mathrm{reg}}}(V(\mathcal{N}_{\mathbb{E}}), \gamma k) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)\ ,\\
\mathbb{J}_1 &:= \mathrm{shadow}_{G_{\mathrm{reg}}}(V(G) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B), \gamma k) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)\ ,\\
\mathbb{J} &:= (\mathbb{XA} \setminus \mathbb{YA}) \cup ((\mathbb{XA} \cup \mathbb{XB}) \setminus \mathbb{YB}) \cup V_{\rightsquigarrow\mathbb{H}} \cup L_\# \cup \mathbb{J}_1\\
&\quad \cup \mathrm{shadow}_{G_{\mathcal{D}} \cup G_\nabla}(V_{\rightsquigarrow\mathbb{H}} \cup L_\# \cup \mathbb{J}_{\mathbb{E}} \cup \mathbb{J}_1, \eta^2 k/10^5)\ ,\\
\mathbb{J}_2 &:= \mathbb{XA} \cap \mathrm{shadow}_{G_\nabla}(S^0 \setminus V(\mathcal{M}_A), \sqrt{\gamma} k)\ ,\\
\mathbb{J}_3 &:= \mathbb{XA} \cap \mathrm{shadow}_{G_\nabla}(\mathbb{XA}, \eta^3 k/10^3)\ .
\end{aligned}$$

Eliminating $\mathbb{J}_{\mathbb{E}}$ from an embedding procedure, for example, will guarantee that we will not be forced to enter the set $\mathcal{N}_{\mathbb{E}}$. This is convenient in some situations. Which sets are "bad" depends on the particular configuration we want to get. That is, some properties given in the definitions of our configurations in section 4.1 *could be phrased in terms of avoiding some of the sets $\mathbb{J}_{\mathbb{E}}$, $\mathbb{J}_1$, $\mathbb{J}$, $\mathbb{J}_2$, and $\mathbb{J}_3$. For some other properties of the configurations, we take only some of the sets $\mathbb{J}_{\mathbb{E}}$, $\mathbb{J}_1$, $\mathbb{J}$, $\mathbb{J}_2$, and $\mathbb{J}_3$ as initial natural forbidden sets, but then we need to apply some nontrivial cleaning (in Lemmas* 6.1–6.3*) to get a desired configuration.*

We define a set $\mathcal{F}$ of clusters of $\mathcal{M}_A \cup \mathcal{M}_B$,

$$\mathcal{F} := \{C \in \mathcal{V}(\mathcal{M}_A) : C \subseteq \mathbb{XA}\} \cup \mathcal{V}_1(\mathcal{M}_B)\ . \tag{3.14}$$

As it turns out (see Lemma 3.11*), $\mathcal{F}$ is actually an $(\mathcal{M}_A \cup \mathcal{M}_B)$-cover.*

On the interface between Lemma 4.17 and Lemma 6.3 we shall need to work with a regularized matching which is formed of only those edges $E(\mathcal{D})$ which are either incident to $\mathbb{E}$ or included in G_{reg}. The following lemma provides us with an appropriate "cleaned version of $\mathcal{D}$." The notion of being absorbed adapts in a straightforward way to two families of dense spots: A family of dense spots $\mathcal{D}_1$ *is absorbed* by another family $\mathcal{D}_2$ if for every $D_1 \in \mathcal{D}_1$ there exists $D_2 \in \mathcal{D}_2$ such that D_1 is contained in D_2 as a subgraph.

LEMMA 3.6. *Assume we are in Setting* 3.5*. Then there exists a family $\mathcal{D}_\nabla$ of edge-disjoint $(\gamma^3 k/4, \gamma/2)$-dense spots absorbed by $\mathcal{D}$ such that*

1. $|E(\mathcal{D}) \setminus E(\mathcal{D}_\nabla)| \leqslant \rho kn$, *and*
2. $E(\mathcal{D}_\nabla) \subseteq E(G_{\mathrm{reg}}) \cup E(G[\mathbb{E}, \mathbb{E} \cup \bigcup \mathbf{V}])$.

The proof of Lemma 3.6 is a warm-up for proofs in section 5.

Proof of Lemma 3.6. Let $\mathcal{D}^- \subseteq \mathcal{D}$ be the set of dense spots $D \in \mathcal{D}$ for which

$$\sqrt{\gamma} e(D) \leqslant \left| E(D) \setminus \Big(E(G_{\mathrm{reg}}) \cup E\Big(G\Big[\mathbb{E}, \mathbb{E} \cup \bigcup \mathbf{V}\Big]\Big)\Big) \right| .$$

Thus,

$$\begin{aligned} \sqrt{\gamma} e(\mathcal{D}^-) &\leqslant \left| E(\mathcal{D}^-) \setminus \Big(E(G_{\mathrm{reg}}) \cup E\Big(G\Big[\mathbb{E}, \mathbb{E} \cup \bigcup \mathbf{V}\Big]\Big)\Big) \right| \\ &\leqslant \left| E(\mathcal{D}) \setminus \Big(E(G_{\mathrm{reg}}) \cup E\Big(G\Big[\mathbb{E}, \mathbb{E} \cup \bigcup \mathbf{V}\Big]\Big)\Big) \right| \\ {\scriptstyle (\text{by S3.5(10)})} \quad &\leqslant \frac{5}{4}\gamma kn\ . \end{aligned} \tag{3.15}$$

For each $D \in \mathcal{D} \setminus \mathcal{D}^-$ we show below how to extract a $(\gamma^3 k/4, \gamma/2)$-dense spot $D' \subseteq D$ with

$$e(D') \geqslant (1 - 2\sqrt{\gamma}) e(D) \tag{3.16}$$

and $E(D') \subseteq E(G_{\mathrm{reg}}) \cup E(G[\mathbb{E}, \mathbb{E} \cup \bigcup \mathbf{V}])$. Let $\mathcal{D}_\nabla$ be the set of all D' obtained in this way. That is, we have $E(\mathcal{D}_\nabla) \subseteq E(\mathcal{D} \setminus \mathcal{D}^-)$. This ensures property 2. We also have property 1, since

$$\begin{aligned} |E(\mathcal{D}) \setminus E(\mathcal{D}_\nabla)| &= |E(\mathcal{D}^-)| + |E(\mathcal{D} \setminus \mathcal{D}^-) \setminus E(\mathcal{D}_\nabla)| \\ {\scriptstyle ((3.15)\text{ for 1st term and }(3.16)\text{ for 2nd term})} \quad &\leqslant \frac{5}{4}\sqrt{\gamma} kn + 2\sqrt{\gamma} \cdot e(\mathcal{D}) \\ {\scriptstyle (\text{as } e(\mathcal{D}) \leqslant e(G) \leqslant kn)} \quad &\leqslant \rho kn\ . \end{aligned}$$

We now show how to extract a $(\gamma^3 k/4, \gamma/2)$-dense spot $D' \subseteq D$ with $e(D') \geqslant (1-2\sqrt{\gamma})e(D)$ and $E(D') \subseteq E(G_{\mathrm{reg}}) \cup E(G[\mathbb{E}, \mathbb{E} \cup \bigcup \mathbf{V}])$ from any spot $D \in \mathcal{D} \setminus \mathcal{D}^-$. Let $D = (A, B; F)$, and let $a := |A|$, $b := |B|$. As D is $(\gamma k, \gamma)$-dense, we have $a, b \geqslant \gamma k$. Note also that Definition 2.5 gives that

$$e(D) \geqslant \gamma ab > \frac{\gamma^{1.5} ab}{2} \ . \tag{3.17}$$

First, we discard from D all edges not contained in $E(G_{\mathrm{reg}}) \cup E(G[\mathbb{E}, \mathbb{E} \cup \bigcup \mathbf{V}])$ to obtain a dense spot $D^* \subseteq D$ with $e(D^*) \geqslant (1-\sqrt{\gamma})e(D)$. Next, we perform a sequential cleaning procedure in D^*. As long as there are such vertices, discard from A any vertex whose current degree is less than $\gamma^2 b/4$, and discard from B any vertex whose current degree is less than $\gamma^2 a/4$. When this procedure terminates, the resulting graph $D' = (A', B'; F')$ has $\mathrm{mindeg}_{D'}(A') \geqslant \gamma^2 b/4 \geqslant \gamma^3 k/4$ and $\mathrm{mindeg}_{D'}(B') \geqslant \gamma^3 k/4$. Note that we deleted at most $a \cdot \gamma^2 b/4 + b \cdot \gamma^2 a/4$ edges out of the at least $(1-\sqrt{\gamma})e(D)$ edges of D^*. This means that

$$e(D') \geqslant (1-\sqrt{\gamma})e(D) - \gamma^2 ab/2 \overset{(3.17)}{\geqslant} (1-2\sqrt{\gamma})e(D) \ ,$$

as desired. Thus we also have the required density of D', namely $\mathrm{d}_{D'}(A', B') \geqslant (1-2\sqrt{\gamma})\gamma \geqslant \gamma/2$. ◻

In some cases we shall also partition the set $V(G)$ into three sets as in Lemma 3.3. This motivates the following definition.

DEFINITION 3.7 (proportional splitting). *Let $\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{p}_2 > 0$ be three positive reals with $\sum_i \mathfrak{p}_i \leqslant 1$. Under Setting* 3.5*, suppose that $(\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2)$ is a partition of $V(G) \setminus \mathbb{H}$ satisfying the assertions of Lemma* 3.3 *with parameter $p_{\mathrm{L3.3}} := 10$ for graph $G^*_{\mathrm{L3.3}} := (G_\nabla - \mathbb{H}) \cup G_{\mathcal{D}}$ (here the union means union of the edges), bounded decomposition $(\mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathbb{E})$, matching $\mathcal{M}_{\mathrm{L3.3}} := \mathcal{M}_A \cup \mathcal{M}_B$, sets $\mathbb{B}_1 := V_{\mathrm{good}}, \mathbb{B}_2 := \mathbb{XA} \setminus (\mathbb{H} \cup \mathbb{J})$, $\mathbb{B}_3 := \mathbb{XB} \setminus \mathbb{J}$, $\mathbb{B}_4 := V(G_{\mathrm{exp}})$, $\mathbb{B}_5 := \mathbb{E}$, $\mathbb{B}_6 := V_{\rightsquigarrow\mathbb{E}}$, $\mathbb{B}_7 := \mathbb{J}_{\mathbb{E}}$, $\mathbb{B}_8 := \mathbb{L}_{\eta,k}(G)$, $\mathbb{B}_9 := L_\#$, $\mathbb{B}_{10} := V_{\rightsquigarrow\mathbb{H}}$, and reals $\mathfrak{q}_1 := \mathfrak{p}_0, \mathfrak{q}_2 := \mathfrak{p}_1$, $\mathfrak{q}_3 := \mathfrak{p}_2$, $\mathfrak{q}_4 := \cdots = \mathfrak{q}_{10} = 0$. Note that by Lemma* 3.3(8) *we have that $(\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2)$ is a partition of $V(G) \setminus \mathbb{H}$. We call $(\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2)$ a* proportional $(\mathfrak{p}_0 : \mathfrak{p}_1 : \mathfrak{p}_2)$ splitting.

We refer to properties of the proportional $(\mathfrak{p}_0 : \mathfrak{p}_1 : \mathfrak{p}_2)$ splitting $(\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2)$ using the numbering of Lemma 3.3*; for example, "Definition* 3.7(5)*" tells us, among other things, that $|(\mathbb{XA} \setminus (\mathbb{J} \cup \mathbb{H})) \cap \mathbb{A}_0| \geqslant \mathfrak{p}_0 |\mathbb{XA} \setminus (\mathbb{J} \cup \mathbb{H})| - n^{0.9}$.*

SETTING 3.8. *Under Setting* 3.5*, suppose that we are given a proportional $(\mathfrak{p}_0 : \mathfrak{p}_1 : \mathfrak{p}_2)$ splitting $(\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2)$ of $V(G) \setminus \mathbb{H}$. We assume that*

$$\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{p}_2 \geqslant \frac{\eta}{100} \ . \tag{3.18}$$

Let $\bar{V}, \bar{\mathcal{V}}, \bar{\mathbf{V}}$ be the exceptional sets as in Definition 3.7(1)*.*

We write

$$\mathbb{F} := \mathrm{shadow}_{G_{\mathcal{D}}}\left(\bigcup \bar{\mathcal{V}} \cup \bigcup \bar{\mathcal{V}}^* \cup \bigcup \bar{\mathbf{V}}, \frac{\eta^2 k}{10^{10}}\right) \ , \tag{3.19}$$

where $\bar{\mathcal{V}}^*$ are a family of partners of $\bar{\mathcal{V}}$ in $\mathcal{M}_A \cup \mathcal{M}_B$.

We have

$$|\mathbb{F}| \leqslant \varepsilon n \ . \tag{3.20}$$

For an arbitrary set $U \subseteq V(G)$ and for $i \in \{0,1,2\}$, we write $U^{\restriction i}$ for the set $U \cap \mathbb{A}_i$.

For each $(X,Y) \in \mathcal{M}_A \cup \mathcal{M}_B$ such that $X, Y \notin \bar{\mathcal{V}}$, we write $(X,Y)^{\restriction i}$ for an arbitrary fixed pair $(X' \subseteq X, Y' \subseteq Y)$ with the property that $|X'| = |Y'| = \min\{|X^{\restriction i}|, |Y^{\restriction i}|\}$. We extend this notion of restriction to an arbitrary regularized matching $\mathcal{N} \subseteq \mathcal{M}_A \cup \mathcal{M}_B$ as follows. We set

$$\mathcal{N}^{\restriction i} := \left\{(X,Y)^{\restriction i} \ : \ (X,Y) \in \mathcal{N} \text{ with } X, Y \notin \bar{\mathcal{V}}\right\} .$$

The next lemma provides some simple properties of a restriction of a regularized matching.

Lemma 3.9. *Assume Settings* 3.5 *and* 3.8*. Then for each $i \in \{0,1,2\}$ and for each $\mathcal{N} \subseteq \mathcal{M}_A \cup \mathcal{M}_B$, we have that $\mathcal{N}^{\restriction i}$ is a $(\frac{400\varepsilon}{\eta}, \frac{d}{2}, \frac{\eta\pi}{200}\mathfrak{c})$-regularized matching satisfying*

$$|V(\mathcal{N}^{\restriction i})| \geqslant \mathfrak{p}_i |V(\mathcal{N})| - 2k^{-0.05} n \ . \tag{3.21}$$

Moreover, for all $v \notin \mathbb{F}$ and for all $i = 0,1,2$, we have $\deg_{G_{\mathcal{D}}}(v, V(\mathcal{N})^{\restriction i} \setminus V(\mathcal{N}^{\restriction i})) \leqslant \frac{\eta^2 k}{10^5}$.

Proof. Let us consider an arbitrary pair $(X,Y) \in \mathcal{N}$. By Definition 3.7(3) we have

$$|X^{\restriction i}| \geqslant \mathfrak{p}_i|X| - k^{0.9} \overset{(3.18)}{\geqslant} \frac{\eta}{200}|X| \quad \text{and} \quad |Y^{\restriction i}| \geqslant \mathfrak{p}_i|Y| - k^{0.9} \overset{(3.18)}{\geqslant} \frac{\eta}{200}|Y| \ . \tag{3.22}$$

In particular, Fact 2.1 gives that $(X,Y)^{\restriction i}$ is a $400\varepsilon/\eta$-regular pair of density at least $d/2$.

We now turn to (3.21). The total order of pairs $(X,Y) \in \mathcal{N}$ excluded entirely from $\mathcal{N}^{\restriction i}$ is at most

$$2\exp(-k^{0.1})n < k^{-0.05}n \tag{3.23}$$

by Definition 3.7(1). Further, for each $(X,Y) \in \mathcal{N}$ whose part is included in $\mathcal{N}^{\restriction i}$ we have that

$$|V((X,Y)^{\restriction i})| \overset{(3.22)}{\geqslant} \mathfrak{p}_i(|X| + |Y|) - 2k^{0.9} \ . \tag{3.24}$$

Recall that $\mathcal{M}_A$ and $\mathcal{M}_B$ are $(\varepsilon, d, \pi\mathfrak{c})$-regularized. In particular, $\mathcal{M}_A$ and $\mathcal{M}_B$ are $(\varepsilon, d, k^{0.95})$-regularized. Consequently,

$$|\mathcal{N}| \leqslant |\mathcal{M}_A \cup \mathcal{M}_B| \leqslant \frac{n}{2k^{0.95}} \ . \tag{3.25}$$

Collecting the loss caused by entirely excluded pairs in (3.23) and the loss of at most $2k^{0.9}$ vertices from (3.24) to each of the at most $|\mathcal{N}|$-many nonexcluded pairs, we get that

$$|V(\mathcal{N}^{\restriction i})| \overset{(3.23)}{\geqslant} \mathfrak{p}_i|V(\mathcal{N})| - k^{-0.05}n - 2k^{0.9}|\mathcal{N}| \overset{(3.25)}{\geqslant} \mathfrak{p}_i|V(\mathcal{N})| - 2k^{-0.05}n \ ,$$

and (3.21) follows.

For the "moreover" part of the lemma, note that by Facts 2.7 and 2.8

$$\deg_{G_{\mathcal{D}}}(v, V(\mathcal{N})^{\restriction i} \setminus V(\mathcal{N}^{\restriction i})) \leqslant \frac{\eta^2 k}{10^{10}} + \frac{(\Omega^*)^2}{\pi\nu\gamma^2} \cdot 3k^{0.9} \leqslant \frac{\eta^2 k}{10^5} \ .$$

This completes the proof. ∎

The following lemma gives a useful bound on the sizes of some sets defined in Setting 3.5.

LEMMA 3.10. *Suppose we are in Setting* 3.5. *Let*

$$\beta > \eta^2\sqrt{\gamma} \tag{3.26}$$

be arbitrary. Suppose that all but at most βkn edges are captured by ∇. Then,

$$|L_\#| \leqslant \frac{20\beta}{\eta}n\ , \tag{3.27}$$

$$|\mathbb{XA}\setminus\mathbb{YA}| \leqslant \frac{600\beta}{\eta^2}n\ , \tag{3.28}$$

$$|(\mathbb{XA}\cup\mathbb{XB})\setminus\mathbb{YB}| \leqslant \frac{600\beta}{\eta^2}n\ . \tag{3.29}$$

Further, let $\tilde\beta > 0$ be arbitrary. If $e_G(\mathbb{H},\mathbb{XA}\cup\mathbb{XB}) \leqslant \tilde\beta kn$, then

$$|V_{\leadsto\mathbb{H}}| \leqslant \frac{100\tilde\beta n}{\eta}\ . \tag{3.30}$$

Proof. Let $W_1 := \{v \in V(G) \,:\, \deg_G(v) - \deg_{G_\nabla}(v) \geqslant \eta k/100\}$. We have $|W_1| \leqslant \frac{200\beta}{\eta}n \leqslant \frac{100\beta}{\eta^2}n$.

Observe that $L_\#$ sends out at most $(1+\frac{9}{10}\eta)k|L_\#| < \frac{40\beta}{\eta}kn$ edges in G_∇. Let $W_2 := \{v \in V(G) \,:\, \deg_{G_\nabla}(v,L_\#) \geqslant \eta k/10\}$. We have $|W_2| \leqslant \frac{400\beta}{\eta^2}n$.

Let $W_3 := \{v \in \mathbb{XA} \,:\, \deg_{G_\nabla}(v, S^0\setminus V(\mathcal{M}_A)) \geqslant \sqrt{\gamma}k\}$. By Setting 3.5(6) we have

$$|W_3| \leqslant \sqrt{\gamma}n \overset{(3.26)}{\leqslant} \frac{\beta}{\eta^2}n\ .$$

For (3.28), observe that $\mathbb{XA}\setminus\mathbb{YA} \subseteq W_1\cup W_2\cup W_3$. For (3.29), observe that $\mathbb{XB}\setminus\mathbb{YB} \subseteq W_1\cup W_2$ and that $\mathbb{YA}\subseteq\mathbb{YB}$. Thus, $(\mathbb{XA}\cup\mathbb{XB})\setminus\mathbb{YB} \subseteq (\mathbb{XA}\setminus\mathbb{YA})\cup(\mathbb{XB}\setminus\mathbb{YB}) \subseteq W_1\cup W_2\cup W_3$.

The bound (3.30) follows from (3.13). □

We finish this section with an auxiliary result which will only be used later in the proofs of Lemmas 6.2 and 6.3.

LEMMA 3.11. *Assume Settings* 3.5 *and* 3.8. *We have*

$$\mathbb{XA}^{\restriction 0}\setminus(\mathbb{J}\cup\mathbb{F}) \subseteq \mathbb{A}_0\setminus\left(\mathbb{F}\cup\mathrm{shadow}_{G_{\mathcal{D}}}\left(V_{\leadsto\mathbb{H}},\frac{\eta^2k}{10^5}\right)\right), \tag{3.31}$$

$$\mathrm{maxdeg}_{G_\nabla}\left(\mathbb{XA}\setminus(\mathbb{J}_2\cup\mathbb{J}_3),\bigcup\mathcal{F}\right) \leqslant \frac{3\eta^3}{2\cdot 10^3}k\ , \tag{3.32}$$

and for $i = 1,2$ we have

$$\mathrm{mindeg}_{G_\nabla}\left(\mathbb{XA}\setminus(\mathbb{J}\cup\bar V),V_{\mathrm{good}}^{\restriction i}\right) \geqslant \mathfrak{p}_i\left(1+\frac{\eta}{20}\right)k\ , \tag{3.33}$$

$$\mathrm{mindeg}_{G_\nabla}\left(\mathbb{XB}\setminus(\mathbb{J}\cup\bar V),V_{\mathrm{good}}^{\restriction i}\right) \geqslant \mathfrak{p}_i\left(1+\frac{\eta}{20}\right)\frac{k}{2}\ . \tag{3.34}$$

Moreover, $\mathcal{F}$ defined in (3.14) *is an $(\mathcal{M}_A\cup\mathcal{M}_B)$-cover.*

Proof. The definition of $\mathbb{J}$ gives (3.31).

For (3.33) and (3.34), assume that $i=2$ (the other case is analogous). Observe that

$$\begin{aligned}
&\mathrm{mindeg}_{G_\nabla}\left(\mathbb{YA}\setminus(V_{\rightsquigarrow\mathbb{H}}\cup\bar{V}), V_{\mathrm{good}}^{\upharpoonright 2}\right)\\
{\scriptstyle(\text{by D3.7(6)})}\quad &\geqslant \mathfrak{p}_2\cdot \mathrm{mindeg}_{G_\nabla}(\mathbb{YA}\setminus V_{\rightsquigarrow\mathbb{H}}, V_{\mathrm{good}}) - k^{0.9}\\
{\scriptstyle(\text{by (3.10)})}\quad &\geqslant \mathfrak{p}_2\cdot\left(\mathrm{mindeg}_{G_\nabla}(\mathbb{YA}, V_+\setminus L_\#) - \mathrm{maxdeg}_{G_\nabla}(\mathbb{YA}\setminus V_{\rightsquigarrow\mathbb{H}}, \mathbb{H})\right) - k^{0.9}\\
{\scriptstyle(\text{by (3.11), (3.13)})}\quad &\geqslant \mathfrak{p}_2\cdot\left(\left(1+\frac{\eta}{10}\right)k - \frac{\eta k}{100}\right) - k^{0.9}\\
{\scriptstyle(\text{by (3.4), (3.18)})}\quad &\geqslant \mathfrak{p}_2\cdot\left(1+\frac{\eta}{20}\right)k\,,
\end{aligned}$$

which proves (3.33), as $\mathbb{XA}\setminus(\mathbb{J}\cup\bar{V})\subseteq\mathbb{YA}\setminus(V_{\rightsquigarrow\mathbb{H}}\cup\bar{V})$. Similarly, we obtain that

$$\mathrm{mindeg}_{G_\nabla}\left(\mathbb{YB}\setminus(V_{\rightsquigarrow\mathbb{H}}\cup\bar{V}), V_{\mathrm{good}}^{\upharpoonright 2}\right)\geqslant \mathfrak{p}_2\left(1+\frac{\eta}{20}\right)\frac{k}{2}\,,$$

which proves (3.34).

We have $\mathrm{maxdeg}_{G_\nabla}(\mathbb{XA}\setminus\mathbb{J}_3,\mathbb{XA}) < \frac{\eta^3}{10^3}k$ and $\mathrm{maxdeg}_{G_\nabla}(\mathbb{XA}\setminus\mathbb{J}_2, S^0\setminus V(\mathcal{M}_A)) < \sqrt{\gamma}k$. Thus (3.32) follows from Setting 3.5(2) and by (3.4).

For the "moreover" part, it suffices to prove that $\{C\in\mathcal{V}(\mathcal{M}_A): C\subseteq\mathbb{XA}\} = \mathcal{F}\setminus\mathcal{V}_1(\mathcal{M}_B)$ is an $\mathcal{M}_A$-cover. Let $(T_1,T_2)\subseteq\mathcal{M}_A$. As $G\in\mathbf{LKSsmall}(n,k,\eta)$, we have by Setting 3.5(3) that for some $i\in\{1,2\}$, T_i is contained in $\mathbb{L}_{\eta,k}(G)$. Then by Setting 3.5(1), $T_i\subseteq\mathbb{XA}$, as desired. □

4. Ten types of configurations. We now come to the heart of the present paper. We will introduce ten configurations—denoted ($\diamond$**1**)–($\diamond$**10**)—which may be found in a graph $G\in\mathbf{LKS}(n,k,\eta)$.[5] We will be able to infer from the main results of this section (Lemmas 6.1–6.3) and from other structural results of this paper and of [HKP$^+$b] that each graph $G\in\mathbf{LKS}(n,k,\eta)$ contains at least one of these configurations. Lemmas 6.1–6.3 are based on the structure provided by [HKP$^+$b, Lemma 5.4]. We refer the reader to [HKP$^+$d, section 6.1], where we describe in more detail how each of the configurations ($\diamond$**1**)–($\diamond$**10**) can be used for the embedding of any given tree from $\mathbf{trees}(k)$, as required for Theorem 1.2. A full description and proofs of the embedding strategies are given in [HKP$^+$d, section 6.5].

The organization of this section is as follows. In section 4.1, we state some preliminary definitions and introduce the configurations ($\diamond$**1**)–($\diamond$**10**). In section 5, we prove certain "cleaning lemmas." The main results are then stated and proved in section 6. The results of section 6 rely on the auxiliary lemmas of section 3.2 and 5.

4.1. The configurations. We can now define the preconfigurations (**♣**), (♡**1**), (♡**2**), (**exp**), and (**reg**), as well as the configurations[6] ($\diamond$**1**)–($\diamond$**10**). Lemma 4.17 (the proof of which occupies section 6) asserts that each graph $\mathbf{LKS}(n,k,\eta)$ contains at least one of the configurations ($\diamond$**1**)–($\diamond$**10**). More precisely, after getting the "rough structure" we obtained in [HKP$^+$b], we get one of the configurations ($\diamond$**1**)–($\diamond$**10**) from Lemma 4.17, which builds on the analysis given in Lemmas 6.1–6.3.

We now give a brief overview of these configurations. Recall that for our proof of Theorem 1.2 we combine these configurations (in the host graph $G_{\mathrm{T1.2}}$) with a given fine partition of the tree $T_{\mathrm{T1.2}}$ which was informally explained in section 2.6.

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[5] Saying that "we have configuration X," "the graph is in configuration X," or "configuration X occurs" is the same.

[6] The word "configuration" is used for a final structure in a graph which is suitable for embedding purposes, while "preconfigurations" are building blocks for configurations.

Configuration ($\diamond$**1**) covers the easy and lucky case when G contains a subgraph with high minimum degree. A very simple tree-embedding strategy similar to the greedy strategy turns out to work in this case.

The purpose of preconfiguration ($\clubsuit$) is to utilize vertices of $\mathbb{H}$. On the one hand, these vertices seem very powerful because of their large degree; on the other hand, the edges incident to them are very unstructured. Therefore preconfiguration ($\clubsuit$) distills some structure in $\mathbb{H}$. This preconfiguration is then a part of configurations ($\diamond$**2**)–($\diamond$**5**) which deal with the case when $\mathbb{H}$ is substantial. Indeed, Lemma 6.1 asserts that whenever $\mathbb{H}$ is incident to many edges, then at least one of configurations ($\diamond$**1**)–($\diamond$**5**) must occur.

Let us note that each of the configurations ($\diamond$**1**)–($\diamond$**5**) alone suffices for embedding all k-vertex trees. However, when $\mathbb{H}$ is negligible, we may need different configurations ($\diamond$**6**)–($\diamond$**10**) (with different parameters) for embedding different individual trees from **trees**(k).

The cases when the number of edges incident to $\mathbb{H}$ is negligible are covered by configurations ($\diamond$**6**)–($\diamond$**10**). More precisely, in this setting Lemma 4.17 transforms the output structure we obtained in [HKP$^+$b] into an input structure for either Lemma 6.2 or Lemma 6.3. These lemmas then assert that, indeed, one of the configurations ($\diamond$**6**)–($\diamond$**10**) must occur. The configurations ($\diamond$**6**)–($\diamond$**8**) involve combinations of one of the two preconfigurations ($\heartsuit$**1**) and ($\heartsuit$**2**) and one of the two preconfigurations (**exp**) and (**reg**). The idea here is that the hubs are embedded using the structure of (**exp**) or (**reg**) (whichever is applicable), the internal shrubs are embedded using the structure which is specific to each of the configurations ($\diamond$**6**)–($\diamond$**8**), and the end shrubs are embedded using the structure of ($\heartsuit$**1**) or ($\heartsuit$**2**). For this reason, configurations ($\diamond$**6**)–($\diamond$**9**) are accompanied by parameters (denoted by h, h_1, and h_2 in Definitions 4.11–4.14) which correspond to the total orders of shrubs of different kinds. Configuration ($\diamond$**10**) is very similar to the structures obtained in the dense setting in [PS12, HP16]. Configuration ($\diamond$**9**) should be considered halfway towards the dense setting.

Some of the configurations below are accompanied with parameters in the parentheses; note that we do not make explicit those numerical parameters which are inherited from Setting 3.5.

We start by defining configuration ($\diamond$**1**). This is a very easy configuration in which a modification of the greedy tree-embedding strategy works.

Definition 4.1 (configuration ($\diamond$**1**)). *We say that a graph G is in* configuration ($\diamond$**1**) *if there exists a nonempty bipartite graph $H \subseteq G$ with* $\mathrm{mindeg}_G(V(H)) \geqslant k$ *and* $\mathrm{mindeg}(H) \geqslant k/2$.

We now introduce configurations ($\diamond$**2**)–($\diamond$**5**), which make use of the set $\mathbb{H}$. These configurations build on preconfiguration ($\clubsuit$).

Definition 4.2 (preconfiguration ($\clubsuit$)). *Suppose that we are in Setting* 3.5. *We say that the graph G is in* preconfiguration ($\clubsuit$)($\Omega^\star$) *if the following conditions are satisfied: G contains nonempty sets $L'' \subseteq L' \subseteq \mathbb{L}_{\frac{9}{10}\eta,k}(G_\nabla) \setminus \mathbb{H}$, and a nonempty set $\mathbb{H}' \subseteq \mathbb{H}$ such that*

$$\mathrm{maxdeg}_{G_\nabla}(L', \mathbb{H} \setminus \mathbb{H}') < \frac{\eta k}{100}\ , \tag{4.1}$$

$$\mathrm{mindeg}_{G_\nabla}(\mathbb{H}', L') \geqslant \Omega^\star k\ , \tag{4.2}$$

$$\mathrm{maxdeg}_{G_\nabla}(L'', \mathbb{L}_{\frac{9}{10}\eta,k}(G_\nabla) \setminus (\mathbb{H} \cup L')) \leqslant \frac{\eta k}{100}\ . \tag{4.3}$$

Definition 4.3 (configuration ($\diamond$**2**)). *Suppose that we are in Setting* 3.5. *We* *say that the graph G is in* configuration $(\diamond\mathbf{2})(\Omega^\star,\tilde{\Omega},\beta)$ *if the following conditions are satisfied.*

The triple $L'', L', \mathbb{H}'$ witnesses preconfiguration $(\clubsuit)(\Omega^\star)$ in G. There exist a nonempty set $\mathbb{H}'' \subseteq \mathbb{H}'$, a set $V_1 \subseteq V(G_{\mathrm{exp}}) \cap \mathbb{YB} \cap L''$, and a set $V_2 \subseteq V(G_{\mathrm{exp}})$ with the following properties:

$$\begin{aligned}
\mathrm{mindeg}_{G_\nabla}(\mathbb{H}'', V_1) &\geqslant \tilde{\Omega}k \,, \\
\mathrm{mindeg}_{G_\nabla}(V_1, \mathbb{H}'') &\geqslant \beta k \,, \\
\mathrm{mindeg}_{G_{\mathrm{exp}}}(V_1, V_2) &\geqslant \beta k \,, \\
\mathrm{mindeg}_{G_{\mathrm{exp}}}(V_2, V_1) &\geqslant \beta k \,.
\end{aligned}$$

Definition 4.4 (configuration $(\diamond\mathbf{3})$). *Suppose that we are in Setting* 3.5*. We say that the graph G is in* configuration $(\diamond\mathbf{3})(\Omega^\star,\tilde{\Omega},\zeta,\delta)$ *if the following conditions are satisfied.*

The triple $L'', L', \mathbb{H}'$ witnesses preconfiguration $(\clubsuit)(\Omega^\star)$ in G. There exist a nonempty set $\mathbb{H}'' \subseteq \mathbb{H}'$, a set $V_1 \subseteq \mathbb{E} \cap \mathbb{YB} \cap L''$, and a set $V_2 \subseteq V(G) \setminus \mathbb{H}$ such that the following properties are satisfied:

$$\mathrm{mindeg}_{G_\nabla}(\mathbb{H}'', V_1) \geqslant \tilde{\Omega}k \,,$$

$$\mathrm{mindeg}_{G_\nabla}(V_1, \mathbb{H}'') \geqslant \delta k \,,$$

$$\mathrm{maxdeg}_{G_{\mathcal{D}}}(V_1, V(G) \setminus (V_2 \cup \mathbb{H})) \leqslant \zeta k \,, \tag{4.4}$$

$$\mathrm{mindeg}_{G_{\mathcal{D}}}(V_2, V_1) \geqslant \delta k \,. \tag{4.5}$$

Definition 4.5 (configuration $(\diamond\mathbf{4})$). *Suppose that we are in Setting* 3.5*. We say that the graph G is in* configuration $(\diamond\mathbf{4})(\Omega^\star,\tilde{\Omega},\zeta,\delta)$ *if the following conditions are satisfied.*

The triple $L'', L', \mathbb{H}'$ witnesses preconfiguration $(\clubsuit)(\Omega^\star)$ in G. There exist a nonempty set $\mathbb{H}'' \subseteq \mathbb{H}'$ and sets $V_1 \subseteq \mathbb{YB} \cap L''$, $\mathbb{E}' \subseteq \mathbb{E}$, and $V_2 \subseteq V(G) \setminus \mathbb{H}$ with the following properties:

$$\mathrm{mindeg}_{G_\nabla}(\mathbb{H}'', V_1) \geqslant \tilde{\Omega}k \,,$$

$$\mathrm{mindeg}_{G_\nabla}(V_1, \mathbb{H}'') \geqslant \delta k \,,$$

$$\mathrm{mindeg}_{G_\nabla \cup G_{\mathcal{D}}}(V_1, \mathbb{E}') \geqslant \delta k \,, \tag{4.6}$$

$$\mathrm{mindeg}_{G_\nabla \cup G_{\mathcal{D}}}(\mathbb{E}', V_1) \geqslant \delta k \,, \tag{4.7}$$

$$\mathrm{mindeg}_{G_\nabla \cup G_{\mathcal{D}}}(V_2, \mathbb{E}') \geqslant \delta k \,, \tag{4.8}$$

$$\mathrm{maxdeg}_{G_\nabla \cup G_{\mathcal{D}}}(\mathbb{E}', V(G) \setminus (\mathbb{H} \cup V_2)) \leqslant \zeta k \,. \tag{4.9}$$

Definition 4.6 (configuration $(\diamond\mathbf{5})$). *Suppose that we are in Setting* 3.5*. We say that the graph G is in* configuration $(\diamond\mathbf{5})(\Omega^\star,\tilde{\Omega},\delta,\zeta,\tilde{\pi})$ *if the following conditions are satisfied.*

The triple $L'', L', \mathbb{H}'$ witnesses preconfiguration $(\clubsuit)(\Omega^\star)$ in G. There exist a nonempty set $\mathbb{H}'' \subseteq \mathbb{H}'$ and a set $V_1 \subseteq (\mathbb{YB} \cap L'' \cap \bigcup \mathbf{V}) \setminus V(G_{\mathrm{exp}})$ such that the following conditions are fulfilled:

$$\mathrm{mindeg}_{G_\nabla}(\mathbb{H}'', V_1) \geqslant \tilde{\Omega}k \,, \tag{4.10}$$

$$\mathrm{mindeg}_{G_\nabla}(V_1, \mathbb{H}'') \geqslant \delta k \,, \tag{4.11}$$

$$\mathrm{mindeg}_{G_{\mathrm{reg}}}(V_1) \geqslant \zeta k \,. \tag{4.12}$$

Further, we have

$$C \cap V_1 = \emptyset \quad \text{or} \quad |C \cap V_1| \geqslant \tilde{\pi}|C| \tag{4.13}$$

for every $C \in \mathbf{V}$.

In remains to introduce configurations $(\diamond\mathbf{6})$–$(\diamond\mathbf{10})$. In these configurations the set $\mathbb{H}$ is not utilized. All these configurations make use of Setting 3.8; i.e., the set $V(G) \setminus \mathbb{H}$ is partitioned into three sets $\mathbb{A}_0, \mathbb{A}_1$, and $\mathbb{A}_2$. The purpose of $\mathbb{A}_0, \mathbb{A}_1$, and $\mathbb{A}_2$ is to make it possible to embed the hubs, the internal shrubs, and the end shrubs of $T_{\mathrm{T1.2}}$, respectively. Thus the parameters $\mathfrak{p}_0, \mathfrak{p}_1$, and $\mathfrak{p}_2$ are chosen proportionally to the sizes of these respective parts of $T_{\mathrm{T1.2}}$.

We first introduce the four preconfigurations $(\heartsuit\mathbf{1})$, $(\heartsuit\mathbf{2})$, $(\mathbf{exp})$, and $(\mathbf{reg})$.

An $\mathcal{M}$*-cover* of a regularized matching $\mathcal{M}$ is a family $\mathcal{F} \subseteq \mathcal{V}(\mathcal{M})$ with the property that at least one of the elements S_1 and S_2 is a member of $\mathcal{F}$ for each $(S_1, S_2) \in \mathcal{M}$.

Definition 4.7 (preconfiguration $(\heartsuit\mathbf{1})$). *Suppose that we are in Settings* 3.5 *and* 3.8. *We say that the graph G is in* preconfiguration $(\heartsuit\mathbf{1})(\gamma', h)$ *of $V(G)$ if there are two nonempty sets* $V_0, V_1 \subseteq \mathbb{A}_0 \setminus (\mathbb{F} \cup \mathrm{shadow}_{G_{\mathcal{D}}}(V_{\rightsquigarrow\mathbb{H}}, \frac{\eta^2 k}{10^5}))$ *with the following properties:*

$$\mathrm{mindeg}_{G_\nabla}\left(V_0, V_{\mathrm{good}}^{\restriction 2}\right) \geqslant h/2\ , \tag{4.14}$$

$$\mathrm{mindeg}_{G_\nabla}\left(V_1, V_{\mathrm{good}}^{\restriction 2}\right) \geqslant h\ . \tag{4.15}$$

Further, there is an $(\mathcal{M}_A \cup \mathcal{M}_B)$-cover $\mathcal{F}$ such that

$$\mathrm{maxdeg}_{G_\nabla}\left(V_1, \bigcup \mathcal{F}\right) \leqslant \gamma' k\ . \tag{4.16}$$

Definition 4.8 (preconfiguration $(\heartsuit\mathbf{2})$). *Suppose that we are in Settings* 3.5 *and* 3.8. *We say that the graph G is in* preconfiguration $(\heartsuit\mathbf{2})(h)$ *of $V(G)$ if there are two nonempty sets* $V_0, V_1 \subseteq \mathbb{A}_0 \setminus (\mathbb{F} \cup \mathrm{shadow}_{G_{\mathcal{D}}}(V_{\rightsquigarrow\mathbb{H}}, \frac{\eta^2 k}{10^5}))$ *with the following properties:*

$$\mathrm{mindeg}_{G_\nabla}\left(V_0 \cup V_1, V_{\mathrm{good}}^{\restriction 2}\right) \geqslant h. \tag{4.17}$$

Definition 4.9 (preconfiguration $(\mathbf{exp})$). *Suppose that we are in Settings* 3.5 *and* 3.8. *We say that the graph G is in* preconfiguration $(\mathbf{exp})(\beta)$ *if there are two nonempty sets $V_0, V_1 \subseteq \mathbb{A}_0$ with the following properties:*

$$\mathrm{mindeg}_{G_{\mathrm{exp}}}(V_0, V_1) \geqslant \beta k\ , \tag{4.18}$$

$$\mathrm{mindeg}_{G_{\mathrm{exp}}}(V_1, V_0) \geqslant \beta k\ . \tag{4.19}$$

Definition 4.10 (preconfiguration $(\mathbf{reg})$). *Suppose that we are in Settings* 3.5 *and* 3.8. *We say that the graph G is in* preconfiguration $(\mathbf{reg})(\tilde{\varepsilon}, d', \mu)$ *if there are two nonempty sets $V_0, V_1 \subseteq \mathbb{A}_0$ and a nonempty family of vertex-disjoint $(\tilde{\varepsilon}, d')$-superregular pairs $\{(Q_0^{(j)}, Q_1^{(j)}\}_{j\in\mathcal{Y}}$ (with respect to the edge set $E(G)$) with $V_0 := \bigcup Q_0^{(j)}$ and $V_1 := \bigcup Q_1^{(j)}$ such that*

$$\min\left\{|Q_0^{(j)}|, |Q_1^{(j)}|\right\} \geqslant \mu k\ . \tag{4.20}$$

Definition 4.11 (configuration ($\diamond\mathbf{6}$)). *Suppose that we are in Settings* 3.5 *and* 3.8. *We say that the graph G is in* configuration $(\diamond\mathbf{6})(\delta,\tilde{\varepsilon},d',\mu,\gamma',h_2)$ *if the following conditions are satisfied.*

The vertex sets V_0,V_1 witness preconfiguration $(\mathbf{reg})(\tilde{\varepsilon},d',\mu)$ *or preconfiguration* $(\mathbf{exp})(\delta)$ *and either preconfiguration* $(\heartsuit\mathbf{1})(\gamma',h_2)$ *or preconfiguration* $(\heartsuit\mathbf{2})(h_2)$*. There exist nonempty sets $V_2,V_3\subseteq\mathbb{A}_1$ such that*

$$\mathrm{mindeg}_G(V_1,V_2)\geqslant\delta k\ , \tag{4.21}$$

$$\mathrm{mindeg}_G(V_2,V_1)\geqslant\delta k\ , \tag{4.22}$$

$$\mathrm{mindeg}_{G_{\mathrm{exp}}}(V_2,V_3)\geqslant\delta k\ , \tag{4.23}$$

$$\mathrm{mindeg}_{G_{\mathrm{exp}}}(V_3,V_2)\geqslant\delta k\ . \tag{4.24}$$

Definition 4.12 (configuration ($\diamond\mathbf{7}$)). *Suppose that we are in Settings* 3.5 *and* 3.8. *We say that the graph G is in* configuration $(\diamond\mathbf{7})(\delta,\rho',\tilde{\varepsilon},d',\mu,\gamma',h_2)$ *if the following conditions are satisfied.*

The sets V_0,V_1 witness preconfiguration $(\mathbf{reg})(\tilde{\varepsilon},d',\mu)$ *and either preconfiguration* $(\heartsuit\mathbf{1})(\gamma',h_2)$ *or preconfiguration* $(\heartsuit\mathbf{2})(h_2)$*. There exist nonempty sets $V_2\subseteq\mathbb{E}^{\restriction 1}\setminus\bar{V}$ and $V_3\subseteq\mathbb{A}_1$ such that*

$$\mathrm{mindeg}_G(V_1,V_2)\geqslant\delta k\ , \tag{4.25}$$

$$\mathrm{mindeg}_G(V_2,V_1)\geqslant\delta k\ , \tag{4.26}$$

$$\mathrm{maxdeg}_{G_{\mathcal{D}}}(V_2,\mathbb{A}_1\setminus V_3)<\rho' k\ , \tag{4.27}$$

$$\mathrm{mindeg}_{G_{\mathcal{D}}}(V_3,V_2)\geqslant\delta k\ . \tag{4.28}$$

Definition 4.13 (configuration ($\diamond\mathbf{8}$)). *Suppose that we are in Settings* 3.5 *and* 3.8. *We say that the graph G is in* configuration $(\diamond\mathbf{8})(\delta,\rho',\varepsilon_1,\varepsilon_2,d_1,d_2,\mu_1,\mu_2,h_1,h_2)$ *if the following conditions are satisfied.*

The vertex sets V_0,V_1 witness preconfigurations $(\mathbf{reg})(\varepsilon_2,d_2,\mu_2)$ *and* $(\heartsuit\mathbf{2})(h_2)$*. There exist nonempty sets $V_2\subseteq\mathbb{A}_0$, $V_3,V_4\subseteq\mathbb{A}_1$, $V_3\subseteq\mathbb{E}\setminus\bar{V}$, and an $(\varepsilon_1,d_1,\mu_1 k)$-regularized matching $\mathcal{N}$ absorbed by $(\mathcal{M}_A\cup\mathcal{M}_B)\setminus\mathcal{N}_{\mathbb{E}}$, $V(\mathcal{N})\subseteq\mathbb{A}_1\setminus V_3$, such that*

$$\mathrm{mindeg}_G(V_1,V_2)\geqslant\delta k\ , \tag{4.29}$$

$$\mathrm{mindeg}_G(V_2,V_1)\geqslant\delta k\ , \tag{4.30}$$

$$\mathrm{mindeg}_{G_\nabla}(V_2,V_3)\geqslant\delta k\ , \tag{4.31}$$

$$\mathrm{mindeg}_{G_\nabla}(V_3,V_2)\geqslant\delta k\ , \tag{4.32}$$

$$\mathrm{maxdeg}_{G_{\mathcal{D}}}(V_3,\mathbb{A}_1\setminus V_4)<\rho' k\ , \tag{4.33}$$

$$\mathrm{mindeg}_{G_{\mathcal{D}}}(V_4,V_3)\geqslant\delta k\ , \tag{4.34}$$

$$\deg_{G_{\mathcal{D}}}(v,V_3)+\deg_{G_{\mathrm{reg}}}(v,V(\mathcal{N}))\geqslant h_1\quad \textit{for each } v\in V_2\ . \tag{4.35}$$

Definition 4.14 (configuration ($\diamond\mathbf{9}$)). *Suppose that we are in Settings* 3.5 *and* 3.8. *We say that the graph G is in* configuration $(\diamond\mathbf{9})(\delta,\gamma',h_1,h_2,\varepsilon_1,d_1,\mu_1,\varepsilon_2,d_2,\mu_2)$ *if the following conditions are satisfied.*

The sets V_0,V_1 together with the $(\mathcal{M}_A\cup\mathcal{M}_B)$-cover $\mathcal{F}'$ witness preconfiguration $(\heartsuit\mathbf{1})(\gamma',h_2)$*. There exists an $(\varepsilon_1,d_1,\mu_1 k)$-regularized matching $\mathcal{N}$ absorbed by $\mathcal{M}_A\cup\mathcal{M}_B$, with $V(\mathcal{N})\subseteq\mathbb{A}_1$. Further, there is a family $\{(Q_0^{(j)},Q_1^{(j)})\}_{j\in\mathcal{Y}}$ as in preconfiguration* $(\mathbf{reg})(\varepsilon_2,d_2,\mu_2)$*. There is a set $V_2\subseteq V(\mathcal{N})\setminus\bigcup\mathcal{F}'\subseteq\bigcup\mathbf{V}$ with the following properties:*

$$\mathrm{mindeg}_{G_{\mathcal{D}}}(V_1,V_2)\geqslant h_1\ , \tag{4.36}$$

$$\mathrm{mindeg}_{G_{\mathcal{D}}}(V_2,V_1)\geqslant\delta k\ . \tag{4.37}$$

Our last configuration, configuration $(\diamond\mathbf{10})$, will lead to an embedding very similar to the one in the dense case (as treated in [PS12]; this will be explained in detail in [HKP$^+$d]). To formalize the configuration we need a preliminary definition. We shall generalize the standard concept of a regularity graph (in the context of regular partitions and Szemerédi's regularity lemma) to graphs with clusters whose sizes are bounded only from below.

DEFINITION 4.15 ($(\varepsilon, d, \ell_1, \ell_2)$-regularized graph). *Let G be a graph, and let $\mathcal{V}$ be an ℓ_1-ensemble that partitions $V(G)$. Suppose that $G[X]$ is empty for each $X \in \mathcal{V}$, and suppose $G[X, Y]$ is ε-regular and of density either* 0 *or at least d for each $X, Y \in \mathcal{V}$. Further suppose that for all $X \in \mathcal{V}$ it holds that $|\bigcup \mathrm{N}_G(X)| \leqslant \ell_2$. Then we say that $(G, \mathcal{V})$ is an $(\varepsilon, d, \ell_1, \ell_2)$-*regularized graph.

A regularized matching $\mathcal{M}$ of G is consistent *with $(G, \mathcal{V})$ if $\mathcal{V}(\mathcal{M}) \subseteq \mathcal{V}$.*

DEFINITION 4.16 (configuration $(\diamond\mathbf{10})(\tilde{\varepsilon}, d', \ell_1, \ell_2, \eta')$). *Assume Setting* 3.5. *The graph G contains an $(\tilde{\varepsilon}, d', \ell_1, \ell_2)$-regularized graph $(\tilde{G}, \mathcal{V})$, and there is an $(\tilde{\varepsilon}, d', \ell_1)$-regularized matching $\mathcal{M}$ consistent with $(\tilde{G}, \mathcal{V})$. There are a family $\mathcal{L}^* \subseteq \mathcal{V}$ and distinct clusters $A, B \in \mathcal{V}$ with*

(a) $E(\tilde{G}[A, B]) \neq \emptyset$,

(b) $\deg_{\tilde{G}}(v, V(\mathcal{M}) \cup \bigcup \mathcal{L}^*) \geqslant (1+\eta')k$ *for all but at most $\tilde{\varepsilon}|A|$ vertices $v \in A$ and for all but at most $\tilde{\varepsilon}|B|$ vertices $v \in B$, and*

(c) *for each $X \in \mathcal{L}^*$ we have* $\deg_{\tilde{G}}(v) \geqslant (1+\eta')k$ *for all but at most $\tilde{\varepsilon}|X|$ vertices $v \in X$.*

4.2. The main result. We are now ready to state the main result of the present paper, Lemma 4.17. In the remaining part of the paper we build up the arguments that lead to the proof of Lemma 4.17, which is given in section 6.2.

LEMMA 4.17. *Suppose we are in Settings* 3.5 *and* 3.8. *Further suppose that at least one of the following holds in G:*

(K1) $2e_G(\mathbb{XA}) + e_G(\mathbb{XA}, \mathbb{XB}) \geqslant \eta kn/3$,

(K2) $|V(\mathcal{M}_{\mathrm{good}})| \geqslant \eta n/3$,

where $\mathcal{M}_{\mathrm{good}} := \{(A, B) \in \mathcal{M}_A \ : \ A \cup B \subseteq \mathbb{XA}\}$. Then one of the following configurations occurs in G:

- $(\diamond\mathbf{1})$,
- $(\diamond\mathbf{2})\big(\frac{\eta^{39}\Omega^{**}}{4\cdot 10^{90}(\Omega^*)^{11}}, \frac{\sqrt[4]{\Omega^{**}}}{2}, \frac{\eta^{13}\rho^2}{128\cdot 10^{30}\cdot(\Omega^*)^5}\big)$,
- $(\diamond\mathbf{3})\big(\frac{\eta^{39}\Omega^{**}}{4\cdot 10^{90}(\Omega^*)^{11}}, \frac{\sqrt[4]{\Omega^{**}}}{2}, \frac{\gamma}{2}, \frac{\eta^{13}\gamma^2}{128\cdot 10^{30}\cdot(\Omega^*)^5}\big)$,
- $(\diamond\mathbf{4})\big(\frac{\eta^{39}\Omega^{**}}{4\cdot 10^{90}(\Omega^*)^{11}}, \frac{\sqrt[4]{\Omega^{**}}}{2}, \frac{\gamma}{2}, \frac{\eta^{13}\gamma^3}{384\cdot 10^{30}(\Omega^*)^6}\big)$,
- $(\diamond\mathbf{5})\big(\frac{\eta^{39}\Omega^{**}}{4\cdot 10^{90}(\Omega^*)^{11}}, \frac{\sqrt[4]{\Omega^{**}}}{2}, \frac{\eta^{13}}{128\cdot 10^{30}\cdot(\Omega^*)^3}, \frac{\eta}{2}, \frac{\eta^{13}}{128\cdot 10^{30}\cdot(\Omega^*)^4}\big)$,
- $(\diamond\mathbf{6})\big(\frac{\eta^3\rho^4}{10^{14}(\Omega^*)^4}, 4\pi, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4}, \frac{3\eta^3}{2000}, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)$,
- $(\diamond\mathbf{7})\big(\frac{\eta^3\gamma^3\rho}{10^{12}(\Omega^*)^4}, \frac{\eta\gamma}{400}, 4\pi, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4}, \frac{3\eta^3}{2\cdot 10^3}, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)$,
- $(\diamond\mathbf{8})\big(\frac{\eta^4\gamma^4\rho}{10^{15}(\Omega^*)^5}, \frac{\eta\gamma}{400}, \frac{400\varepsilon}{\eta}, 4\pi, \frac{d}{2}, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta\pi\mathfrak{c}}{200k}, \frac{\eta^2\nu}{2\cdot 10^4}, \mathfrak{p}_1(1+\frac{\eta}{20})k, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)$,
- $(\diamond\mathbf{9})\big(\frac{\rho\eta^8}{10^{27}(\Omega^*)^3}, \frac{2\eta^3}{10^3}, \mathfrak{p}_1(1+\frac{\eta}{40})k, \mathfrak{p}_2(1+\frac{\eta}{20})k, \frac{400\varepsilon}{\eta}, \frac{d}{2}, \frac{\eta\pi\mathfrak{c}}{200k}, 4\pi, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4}\big)$,
- $(\diamond\mathbf{10})\big(\varepsilon, \frac{\gamma^2 d}{2}, \pi\sqrt{\varepsilon'}\nu k, \frac{(\Omega^*)^2 k}{\gamma^2}, \frac{\eta}{40}\big)$.

Remark 4.18. The effect of changing the parameters $\mathfrak{p}_1$ and $\mathfrak{p}_2$ in Setting 3.8 can be more substantial than a mere change of the parameters in one configuration asserted by Lemma 4.17. That is, it may happen that for some values of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ the only configuration that occurs in the graph $G_{\mathrm{L4.17}}$ is, say, $(\diamond\mathbf{6})(\cdot,\cdot,\cdot,\cdot,\cdot,\mathfrak{p}_2(1+$

$\frac{\eta}{20})k)$, while for other values of $\mathfrak{p}_1$ and $\mathfrak{p}_2$, the only configuration that occurs is, say, $(\diamond\mathbf{8})(\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\mathfrak{p}_1(1+\frac{\eta}{20})k,\mathfrak{p}_2(1+\frac{\eta}{20})k)$.

Recall that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are set proportionally to the sizes of the internal and end shrubs of the tree $T_{\mathrm{T1.2}}$, respectively. Thus the above tells us that different trees $T_{\mathrm{T1.2}}$ may be embedded into different parts of $G_{\mathrm{T1.2}}$, and by using different embedding techniques.

Note that it follows from the main results of our previous papers [HKP$^+$a, HKP$^+$b] that graphs from Theorem 1.2 indeed satisfy the hypothesis of Lemma 4.17. More specifically, after obtaining a sparse decomposition of $G_{\mathrm{T1.2}}$ in [HKP$^+$a, Lemma 3.14], we can apply [HKP$^+$b, Lemma 5.4], which asserts that (**K1**) or (**K2**) is fulfilled.

5. Cleaning. This section contains five "cleaning lemmas" (Lemmas 5.1–5.5). The basic setting of all these lemmas is the same. There is a system of vertex sets with some density assumptions on edges between certain sets of this system. The assertion is that a small number of vertices can be discarded from the sets so that some conditions on the minimum degree are fulfilled. While the cleaning strategy is simply discarding the vertices which violate these minimum-degree conditions, the analysis of the outcome is nontrivial. The simplest application of such an approach is the proof of Lemma 3.6 above.

Lemmas 5.1–5.5 are used to get the structures required by the (pre-)configurations introduced in section 4.1.

The first lemma will be used to obtain preconfiguration (♣) in certain situations.

LEMMA 5.1. *Let $\psi\in(0,1)$, and let $\Gamma,\Omega,\Omega'\geqslant 1$ be arbitrary, with*

$$\psi^3\Omega\geqslant 4\Gamma^2\Omega'\ . \tag{5.1}$$

Let P and Q be two disjoint vertex sets in a graph G. Assume that $Y\subseteq V(G)$ is given. We assume that

$$\mathrm{mindeg}(P,Q)\geqslant\Omega k\ , \tag{5.2}$$

$$\mathrm{maxdeg}(Q)\leqslant\Gamma k\ . \tag{5.3}$$

Then there exist sets $P'\subseteq P$, $Q''\subseteq Q'\subseteq Q\setminus Y$ such that the following holds:

(a) $\mathrm{maxdeg}(Q',P\setminus P')<\psi k$,
(b) $\mathrm{maxdeg}(Q'',Q\setminus(Q'\cup Y))<\psi k$,
(c) $\mathrm{mindeg}(P',Q')\geqslant\Omega' k$, *and*
(d) $e(P',Q'')\geqslant(1-\psi)e(P,Q)-|Y\cap Q|\Gamma k$.

Proof. Initially, set $P':=P$, $Q':=Q\setminus Y$, and $Q'':=Q'$. We shall sequentially[7] discard from the sets P', Q', and Q'' those vertices that violate any of properties (a)–(c). Further, if a vertex $v\in Q$ is removed from Q', then we remove it from the set Q'' as well. We thus have $Q''\subseteq Q'$ in each step. After this sequential cleaning procedure finishes it remains only to establish (d).

First, observe that the way we constructed P' (together with (5.2)) ensures that

$$e(P\setminus P',Q'')\leqslant e(P\setminus P',Q')\leqslant\frac{\Omega'}{\Omega}e(P,Q)\ . \tag{5.4}$$

Let $Q^a\subseteq Q$ be the set of the vertices removed from Q' because of condition (a).

[7]No particular order is imposed on the vertices.

Note that a vertex u of $P^c = P \setminus P'$ was removed at some point from the set P' because (c) failed for u. Let C'_u denote the set Q' just before this time. Let $f(u) := \deg(u, C'_u)$. A vertex $v \in Q^a = Q \setminus (Q' \cup Y)$ was removed at some point from the set Q' because (a) failed for v. Let A'_v be the set P' just before this time. Let $g(v) := \deg(v, P \setminus A'_v)$. Observe that $\sum_{u \in P^c} f(u) \geqslant \sum_{v \in Q^a} g(v)$. Indeed, at the moment when $v \in Q$ is removed from Q', the $g(v)$ edges that v sends to the set $P \setminus A'_v$ are counted in $\sum_{u \in \mathrm{N}(v) \cap P^c} f(u)$. Note also that we have $f(u) \leqslant \Omega' k$ and $g(v) \geqslant \psi k$ for each $u \in P^c$ and each $v \in Q^a$, because u and v fail (c) and (a), respectively. We therefore have

$$|P^c|\Omega' k \geqslant \sum_{u \in P^c} f(u) \geqslant \sum_{v \in Q^a} g(v) \geqslant |Q^a|\psi k \ . \tag{5.5}$$

By (5.2) we have

$$|P^c| \leqslant \sum_{u \in P^c} \frac{\deg(u,Q)}{\Omega k} \leqslant \frac{e(P,Q)}{\Omega k} \ . \tag{5.6}$$

Putting (5.5) and (5.6) together, we get that

$$|Q^a| \leqslant \frac{\Omega'}{\psi \Omega k} e(P,Q) \ . \tag{5.7}$$

Because vertices in $Q' \setminus Q''$ fail property (b) we have

$$\begin{aligned} |Q' \setminus Q''|\psi k &\leqslant \sum_{w \in Q' \setminus Q''} \deg(w, Q \setminus (Q' \cup Y)) \overset{(5.3)}{\leqslant} |Q \setminus (Q' \cup Y)|\Gamma k \\ &= |Q^a|\Gamma k \overset{(5.7)}{\leqslant} \frac{\Gamma\Omega'}{\psi\Omega} e(P,Q) \ . \end{aligned} \tag{5.8}$$

Finally, we can lower-bound $e(P', Q'')$ as follows:

$$\begin{aligned} e(P', Q'') &\geqslant e(P,Q) - e(P \setminus P', Q'') - |Y \cap Q|\Gamma k - |Q^a|\Gamma k - |Q' \setminus Q''|\Gamma k \\ \text{(by (5.4), (5.7), (5.8))} \quad &\geqslant e(P,Q)\Big(1 - \frac{\Omega'}{\Omega} - \frac{\Gamma\Omega'}{\psi\Omega} - \frac{\Gamma^2\Omega'}{\psi^2\Omega}\Big) - |Y \cap Q|\Gamma k \\ \text{(by (5.1))} \quad &\geqslant (1-\psi)e(P,Q) - |Y \cap Q|\Gamma k \ . \end{aligned}$$

□

The purpose of the lemmas below (Lemmas 5.2–5.5) is to distill vertex sets for configurations (⋄**2**)–(⋄**10**). They will be applied in Lemmas 6.1–6.3. This is the final “cleaning step” on our way to the proof of Theorem 1.2—the outputs of these lemmas can by used for a vertex-by-vertex embedding of any tree $T \in \mathbf{trees}(k)$ (although the corresponding embedding procedures in [HKP+d] are quite complex).

The first two of these cleaning lemmas (Lemmas 5.2 and 5.3) are suitable when the set $\mathbb{H}$ of vertices of huge degrees (cf. Setting 3.5) needs to be considered.

For the following lemma, recall that we defined $[r]$ as the set of the first r natural numbers, excluding 0.

Lemma 5.2. *For all $r, \Omega^*, \Omega^{**} \in \mathbb{N}$ and $\delta, \gamma, \eta \in (0,1)$, with $(\frac{3\Omega^*}{\gamma})^r \delta < \eta/10$ and $\Omega^{**} > 1000$, the following holds. Suppose there are vertex sets $X_0, X_1, \ldots, X_r$ and Y of an n-vertex graph G such that*

1. $|Y| < \eta n/(4\Omega^*)$,

2. $e(X_0, X_1) \geqslant \eta kn$,
3. $\mathrm{mindeg}(X_0, X_1) \geqslant \Omega^{**}k$,
4. $\mathrm{mindeg}(X_i, X_{i+1}) \geqslant \gamma k$ *for all* $i \in [r-1]$, *and*
5. $\mathrm{maxdeg}(Y \cup \bigcup_{i\in[r]} X_i) \leqslant \Omega^* k$.

Then there are sets $X'_i \subseteq X_i$ *for* $i = 0, 1, \ldots, r$ *such that*

(a) $X'_1 \cap Y = \emptyset$,
(b) $\mathrm{mindeg}(X'_i, X'_{i-1}) \geqslant \delta k$ *for all* $i \in [r]$,
(c) $\mathrm{maxdeg}(X'_i, X_{i+1} \setminus X'_{i+1}) < \gamma k/2$ *for all* $i \in [r-1]$,
(d) $\mathrm{mindeg}(X'_0, X'_1) \geqslant \sqrt{\Omega^{**}}k$, *and*
(e) $e(X'_0, X'_1) \geqslant \eta kn/2$, *in particular* $X'_0 \neq \emptyset$.

Proof. In the formulas below we refer to hypotheses of the lemma as "1."–"5."

Set $X'_1 := X_1 \setminus Y$. For $i = 0, 2, 3, 4, \ldots, r$, set $X'_i := X_i$. Discard sequentially from X'_i any vertex that violates any of the properties (b)–(d). Properties (a)–(d) are trivially satisfied when the procedure terminates. To show that property (e) holds at this point, we bound the number of edges from $e(X_0, X_1)$ that are incident to $X_0 \setminus X'_0$ or with $X_1 \setminus X'_1$ in an amortized way.

For $i \in \{0, \ldots, r\}$ and for $v \in X_i \setminus X'_i$ we write

$$\begin{aligned} f_i(v) &:= \deg\left(v, X_{i+1} \setminus X'_{i+1}(v)\right), \\ g_i(v) &:= \deg\left(v, X'_{i-1}(v)\right), \\ h_i(v) &:= \deg\left(v, X'_{i+1}(v)\right), \end{aligned}$$

where the sets $X'_{i-1}(v), X'_i(v), X'_{i+1}(v)$ above refer to the moment just before v is removed from X'_i (we do not define $f_i(v)$ and $h_i(v)$ for $i = r$ and $g_i(v)$ for $i = 0$).

For $i \in [r]$ let X^b_i denote the vertices in $X_i \setminus X'_i$ that were removed from X'_i because of violating property (b). Then for a given $i \in [r]$ we have that

$$\sum_{v\in X^b_i} g_i(v) < \delta kn . \tag{5.9}$$

For $i = 1, \ldots, r-1$ let X^c_i denote the vertices in $X_i \setminus X'_i$ that violated property (c). Set $X^c_r := \emptyset$. For a given $i \in [r-1]$ we have

$$|X^c_i| \cdot \gamma k/2 \leqslant \sum_{v\in X^c_i} f_i(v) \overset{\text{Fig 3}}{\leqslant} \sum_{w\in X_{i+1}\setminus X'_{i+1}} g_{i+1}(w) \overset{5.,\,(5.9)}{<} \delta kn + |X^c_{i+1}| \cdot \Omega^* k , \tag{5.10}$$

as $X_i \setminus X'_i = X^b_i \cup X^c_i$ for $i = 2, \ldots, r$. Using (5.10) for $j = 0, \ldots, r-1$, we inductively deduce that

$$|X^c_{r-j}|\frac{\gamma}{2} \leqslant \sum_{i=0}^{j-1} \left(\frac{2\Omega^*}{\gamma}\right)^i \delta n . \tag{5.11}$$

(The left-hand side is zero for $j = 0$.) The bound (5.11) for $j = r-1$ gives

$$|X^c_1| \leqslant \frac{2}{\gamma} \cdot \sum_{i=0}^{r-2} \left(\frac{2\Omega^*}{\gamma}\right)^i \delta n \leqslant \frac{2(2\Omega^*)^{r-1}}{\gamma^r}\delta n . \tag{5.12}$$

Therefore,

$$e(X_0, Y \cup X^c_1) \leqslant |Y \cup X^c_1| \cdot \Omega^* k \overset{(5.12),\,1.}{\leqslant} \frac{\eta kn}{4} + \left(\frac{2\Omega^*}{\gamma}\right)^r \delta kn . \tag{5.13}$$

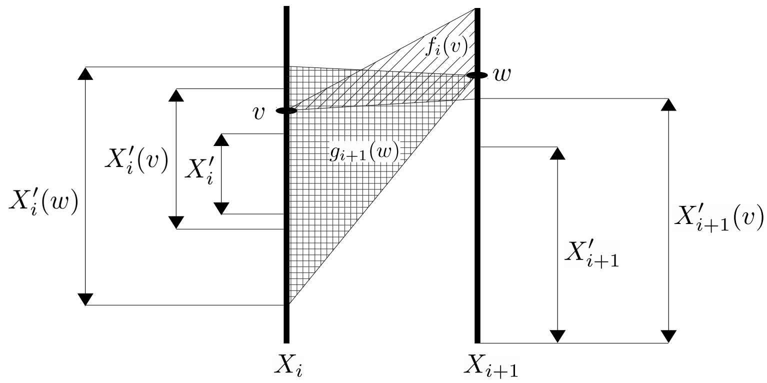

FIG. 3. *Situation in* (5.10). *A summand from* $\sum_{v\in X_i^c} f_i(v)$ *(corresponding edges hatched), and a summand from* $\sum_{w\in X_{i+1}\setminus X'_{i+1}} g_{i+1}(w)$. *Thus the former sum counts the number of edges* vw *such that* $v \in X_i^c$ *and* $w \in X_{i+1} \setminus X'_{i+1}(v)$. *For each such pair* vw *we have that* $v \in X'_i(v) \subseteq X'_i(w)$, *as* w *must have been removed from* X'_{i+1} *prior to* v *being removed from* X'_i. *Hence, the edge* vw *is counted in* $g_{i+1}(w)$ *as well. Similar counting is used in* (5.21) *and in* (5.29).

For any vertex $u \in X_0 \setminus X'_0$ we have $h_0(u) < \sqrt{\Omega^{**}}k$, and at the same time by hypothesis 3 we have $\deg(u, X_1) \geqslant \Omega^{**}k$. So,

$$\sum_{u\in X_0\setminus X'_0} h_0(u) \leqslant \frac{e(X_0,X_1)}{\sqrt{\Omega^{**}}}\ . \tag{5.14}$$

By consulting Figure 4 we have

$$e(X'_0,X'_1) \geqslant e(X_0,X_1) - e(X_0, Y\cup X_1^c) - \sum_{u\in X_0\setminus X'_0} h_0(u) - \sum_{v\in X_1^b} g_1(v)\ . \tag{5.15}$$

Therefore,

$$\begin{aligned} e(X'_0,X'_1) &\geqslant e(X_0,X_1) - e(X_0, Y\cup X_1^c) - \sum_{u\in X_0\setminus X'_0} h_0(u) - \sum_{v\in X_1^b} g_1(v) \\ \text{(by (5.9), (5.13), (5.14))} \quad &\geqslant e(X_0,X_1) - \frac{\eta kn}{4} - \left(\frac{2\Omega^*}{\gamma}\right)^r \delta kn - \frac{e(X_0,X_1)}{\sqrt{\Omega^{**}}} - \delta kn \\ \text{(by 2.)} \quad &\geqslant \eta kn/2\ , \end{aligned}$$

proving property (e). ∎

LEMMA 5.3. *Let* $\delta,\eta,\Omega^*,\Omega^{**},h > 0$, *let* G *be an* n*-vertex graph, let* $X_0, X_1, Y \subseteq V(G)$, *and let* $\mathcal{C}$ *be a family of subsets of* $V(G)$ *such that*

1. $20(\delta + \frac{2}{\sqrt{\Omega^{**}}}) < \eta$,
2. $2kn \geqslant e(X_0,X_1) \geqslant \eta kn$,
3. $\mathrm{mindeg}(X_0,X_1) \geqslant \Omega^{**}k$,
4. $\mathrm{maxdeg}(X_1) \leqslant \Omega^* k$,
5. $|Y| < \eta n/(4\Omega^*)$, *and*
6. $10h|\mathcal{C}|\Omega^* < \eta n$.

Then there are sets $X'_0 \subseteq X_0$ *and* $X'_1 \subseteq X_1 \setminus Y$ *such that*

(a) $\mathrm{mindeg}(X'_0,X'_1) \geqslant \sqrt{\Omega^{**}}k$,

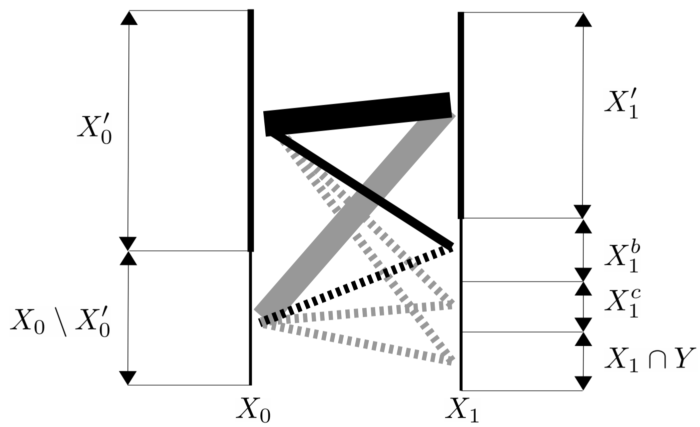

FIG. 4. *The terms in* (5.15). *The edges in the term* $e(X_0, (Y \cap X_1) \cup X_1^c)$ *are shown in dashed gray; some edges of the term* $\sum_{u \in X_0 \setminus X'_0} h_0(u)$ *are shown in thick gray (note that we undercount here, as the summands* $h_0(u)$ *reflect preliminary situations in the set* X'_1*). It is clear that each edge between* X_1^b *and* X'_0 *(thin black) is counted in* $\sum_{v \in X_1^b} g_1(v)$*. Consider now an edge* xv*,* $x \in X_0 \setminus X'_0$*,* $v \in X_1^b$ *(dashed black). Suppose first that* x *was removed from* X'_0 *before* v *was put into* X_1^b*. Then the edge* xv *was counted in* $\sum_{u \in X_0 \setminus X'_0} h_0(u)$*. Suppose next that* v *was put into* X_1^b *before* x *was removed from* X'_0*. Then* xv *was counted in* $\sum_{v \in X_1^b} g_1(v)$*.*

(b) $\mathrm{mindeg}(X'_1, X'_0) \geqslant \delta k$,
(c) *for all* $C \in \mathcal{C}$*, either* $X'_1 \cap C = \emptyset$ *or* $|X'_1 \cap C| \geqslant h$*, and*
(d) $e(X'_0, X'_1) \geqslant \eta kn/2$.

Proof. Set $X'_0 := X_0$ and $X'_1 := X_1 \setminus Y$. Discard sequentially from X'_0 any vertex violating property (a). We discard from X'_1 any vertex violating property (b). Last, we discard from X'_1 all the vertices lying in any set $C \in \mathcal{C}$ violating (c). The deletions from X'_0, or X'_1 can take turns in an arbitrary order until no more are possible. When the process ends, we verify property (d) by bounding the number of edges in $e(X_0, X_1)$ incident to $X_0 \setminus X'_0$ or with $X_1 \setminus X'_1$. Given assumption 2, and since by assumptions 4 and 5 there are at most $\frac{1}{4}\eta kn$ edges incident to $Y \cap X_1$, it suffices to prove that

$$e(X_0, X_1) - e(X'_0, X'_1) - e(Y \cap X_1, X_0) < \frac{\eta kn}{4} \,. \tag{5.16}$$

Denote by X_1^b the set of vertices in $X_1 \setminus (Y \cup X'_1)$ that violated property (b), and by X_1^c the set of vertices in $X_1 \setminus (Y \cup X'_1)$ that violated property (c). Note that for each $C \in \mathcal{C}$, we have $|X_1^c \cap C| < h$, and thus

$$|X_1^c| \leqslant h|\mathcal{C}| \,. \tag{5.17}$$

For a vertex $v \in X_1 \setminus (Y \cup X'_1)$, let $g(v)$ denote $\deg(v, X'_0(v))$, where $X'_0(v)$ denotes the set X'_0 just before v is removed from X'_1. Analogously we define $f(v)$, for $v \in X_0 \setminus X'_0$, as $\deg(v, X'_1(v))$, where the set $X'_1(v)$ denotes the set X'_1 just before v is removed from X_1'. We have

$$\sum_{v\in X_1^b} g(v) < \delta kn,$$

$$\sum_{v\in X_1^c} g(v) \overset{4.}{\leqslant} |X_1^c|\Omega^* k \overset{(5.17)}{\leqslant} h|\mathcal{C}|\cdot\Omega^* k,$$

$$\sum_{v\in X_0\setminus X_0'} f(v) \overset{3.}{\leqslant} \frac{e(X_0,X_1)}{\sqrt{\Omega^{**}}} \overset{2.}{\leqslant} \frac{2}{\sqrt{\Omega^{**}}}kn\ .$$

Thus,

$$\begin{aligned} e(X_0,X_1) - e(X_0',X_1') - e(Y\cap X_1, X_0) &= \sum_{v\in X_1^b} g(v) + \sum_{v\in X_1^c} g(v) + \sum_{v\in X_0\setminus X_0'} f(v) \\ &< \left(\delta + \frac{2}{\sqrt{\Omega^{**}}}\right)kn + h|\mathcal{C}|\Omega^* k \\ \text{(by 1. and 6.)}\quad &< \frac{\eta kn}{4}\ , \end{aligned}$$

establishing (5.16). ◻

The next two lemmas (Lemmas 5.4 and 5.5) deal with cleaning outside the set of huge-degree vertices $\mathbb{H}$.

LEMMA 5.4. *For all $r,\Omega\in\mathbb{N}$, $r\geqslant 2$, and all $\gamma,\delta,\eta>0$ such that*

$$\left(\frac{8\Omega}{\gamma}\right)^r \delta \leqslant \frac{\eta}{10}\ , \tag{5.18}$$

the following holds. Suppose there are vertex sets $Y, X_0, X_1, \dots, X_r \subseteq V$, where V is a set of n vertices. Suppose that edge sets $E_1,\dots,E_r$ are given on V. The expressions deg_i, maxdeg_i, mindeg_i, and e_i below refer to the edge set E_i. Suppose that the following properties are fulfilled:

1. $|Y| < \delta n$.
2. $e_1(X_0,X_1) \geqslant \eta kn$.
3. *For all $i\in[r-1]$ we have* $\mathrm{mindeg}_{i+1}(X_i\setminus Y, X_{i+1}) \geqslant \gamma k$.
4. *For all $i\in\{0,\dots,r-1\}$, we have* $\mathrm{maxdeg}_{i+1}(X_i)\leqslant \Omega k$ *and* $\mathrm{maxdeg}_{i+1}(X_{i+1})\leqslant \Omega k$.

Then there are sets $X_i'\subseteq X_i\setminus Y$ $(i=0,\dots,r)$ satisfying the following:

(a) *For all $i\in[r]$ we have* $\mathrm{mindeg}_i(X_i', X_{i-1}') \geqslant \delta k$.
(b) *For all $i\in[r-1]$ we have* $\mathrm{maxdeg}_{i+1}(X_i', X_{i+1}\setminus X_{i+1}') < \gamma k/2$.
(c) $\mathrm{mindeg}_1(X_0', X_1') \geqslant \delta k$.
(d) $e_1(X_0', X_1') \geqslant \eta kn/2$.

Proof. We proceed similarly as in the proof of Lemma 5.2. Set $X_i' := X_i\setminus Y$ for each $i=0,\dots,r$. Discard sequentially from X_i' any vertex that violates property (a), (b), or (c). When the procedure terminates, we certainly have that (a)–(c) hold. We then show that property (d) holds by bounding the number of edges from $e_1(X_0,X_1)$ that are incident to $X_0\setminus X_0'$ or with $X_1\setminus X_1'$. For $i\in\{0,\dots,r\}$ and for $v\in X_i\setminus X_i'$

we write

$$f_{i+1}(v) := \deg_{i+1}(v, X_{i+1} \setminus X'_{i+1}(v)) ,$$
$$g_i(v) := \deg_i(v, X'_{i-1}(v)) ,$$
$$h(v) := \deg_1(v, X'_1(v)) ,$$

where the sets $X'_1(v)$, $X'_{i-1}(v)$, and $X'_{i+1}(v)$ above refer to the sets X'_1, X'_{i-1}, and X'_{i+1}, respectively, at the moment[8] just before v is removed from X'_i (we do not define $f_{i+1}(v)$ for $i = r$ and $g_i(v)$ for $i = 0$).

Let $X_i^a \subseteq X_i$, $X_i^b \subseteq X_i$ for $i \in [r-1]$ be the sets of vertices removed from X'_i because of properties (a) and (b), respectively. Set $X_r^a := X_r \setminus X'_r$ and $X_0^c := X_0 \setminus X'_0$. We have for each $i \in [r]$

$$\sum_{v \in X_i^a} g_i(v) < \delta kn . \tag{5.19}$$

Also, note that we have

$$\sum_{v \in X_0^c} h(v) \leqslant \delta kn . \tag{5.20}$$

We set $X_r^b := \emptyset$. For a given $i \in [r-1]$ we have

$$\begin{aligned} |X_i^b| \cdot \frac{\gamma k}{2} &\leqslant \sum_{v \in X_i^b} f_{i+1}(v) \\ \text{(see Fig 3)} \quad &\leqslant \sum_{v \in X_{i+1} \setminus X'_{i+1}} g_{i+1}(v) \\ \text{(by 4., (5.19))} \quad &\leqslant \delta kn + |X_{i+1}^b| \Omega k , \end{aligned} \tag{5.21}$$

as $X_i \setminus X'_i \subseteq X_i^a \cup X_i^b \cup Y$ for $i = 2, \ldots, r$. Using (5.21), we deduce inductively that

$$\left| X_{r-j}^b \right| \leqslant \left(\frac{8\Omega}{\gamma} \right)^j \delta n \tag{5.22}$$

for $j = 0, \ldots, r-1$. (The left-hand side is zero for $j = 0$.) Therefore,

$$\begin{aligned} e_1(X'_0, X'_1) &\geqslant e_1(X_0, X_1) - (|Y| + |X_1^b|)\Omega k - \sum_{v \in X_1^a} g_1(v) - \sum_{v \in X_0^c} h(v) \\ \text{(by 2, (5.22), (5.19), (5.20))} \quad &\geqslant \eta kn - \left(\frac{8\Omega}{\gamma} \right)^r \delta kn - 2\delta kn \\ &\geqslant \frac{\eta}{2} kn , \end{aligned}$$

establishing property (d). ◻

Lemma 5.5. *For all $r, \Omega \in \mathbb{N}$, $r \geqslant 2$, and all $\gamma, \eta, \delta, \varepsilon, \mu, d > 0$ with*

$$20\varepsilon < d \quad \text{and} \quad \left(\frac{8\Omega}{\gamma} \right)^r \delta \leqslant \frac{\eta}{30} , \tag{5.23}$$

[8] If $v \in Y$, then this moment is the zeroth step.

the following holds. Suppose there are vertex sets $Y, X_0, X_1, \dots, X_r \subseteq V$*, where* V *is a set of* n *vertices. Let* $P_i^{(1)}, \dots, P_i^{(p)}$ *partition* X_i *for* $i = 0, 1$*. Suppose that edge sets* $E_1, E_2, E_3, \dots, E_r$ *are given on* V*. The expressions* deg_i*,* maxdeg_i*, and* mindeg_i *below refer to the edge set* E_i*. Suppose that*

1. $|Y| < \delta n$,
2. $|X_1| \geqslant \eta n$,
3. *for all* $i \in [r-1]$ *we have* $\mathrm{mindeg}_{i+1}(X_i \setminus Y, X_{i+1}) \geqslant \gamma k$,
4. *the family* $\{(P_0^{(j)}, P_1^{(j)})\}_{j\in[p]}$ *is an* $(\varepsilon, d, \mu k)$*-regularized matching with respect to the edge set* E_1*, and*
5. *for all* $i \in \{0, \dots, r-1\}$ *we have* $\mathrm{maxdeg}_{i+1}(X_{i+1}) \leqslant \Omega k$*, and for all* $i \in \{1, \dots, r-1\}$ *we have* $\mathrm{maxdeg}_{i+1}(X_i) \leqslant \Omega k$.

Then there exist a nonempty family $\mathcal{Y} \subseteq [p]$ *and a family* $\{(Q_0^{(j)}, Q_1^{(j)})\}_{j\in\mathcal{Y}}$ *of vertex-disjoint* $(4\varepsilon, \frac{d}{4})$*-superregular pairs with respect to* E_1*, with*

(a) $|Q_0^{(j)}|, |Q_1^{(j)}| \geqslant \frac{\mu k}{2}$ *for each* $j \in \mathcal{Y}$,

and sets $X'_0 := \bigcup Q_0^{(j)} \subseteq X_0 \setminus Y$*,* $X'_1 := \bigcup Q_1^{(j)} \subseteq X_1 \setminus Y$*,* $X'_i \subseteq X_i \setminus Y$ *(*$i = 2, \dots, r$*) such that*

(b) *for all* $i \in [r-1]$ *we have* $\mathrm{mindeg}_{i+1}(X'_{i+1}, X'_i) \geqslant \delta k$*, and*

(c) *for all* $i \in [r-1]$*, we have* $\mathrm{maxdeg}_{i+1}(X'_i, X_{i+1} \setminus X'_{i+1}) < \gamma k/2$.

Proof. Initially, set $\mathcal{J} := \emptyset$ and $X'_i := X_i \setminus Y$ for each $i = 0, \dots, r$. Discard sequentially from X'_i any vertex that violates one or both of the properties (b) and (c). We would like to keep track of these vertices, and therefore we call $X_i^b, X_i^c \subseteq X_i$ the sets of vertices removed from X'_i because of properties (b) and (c), respectively. Further, for $i = 0, 1$ and for $j \in [p]$ remove any vertex $v \in X'_i \cap P_i^{(j)}$ from X'_i if

$$\mathrm{deg}_1(v, X'_{1-i} \cap P_{1-i}^{(j)}) \leqslant \frac{d|P_{1-i}^{(j)}|}{4} . \tag{5.24}$$

For $i = 0, 1$, let X_i^a be the set of those vertices of X_i that were removed because of (5.24).

If for some $j \in [p]$ we have $|P_0^{(j)} \cap Y| > \frac{|P_0^{(j)}|}{4}$ or $|P_1^{(j)} \cap (Y \cup X_1^c)| > \frac{|P_1^{(j)}|}{4}$, we remove simultaneously the sets $P_0^{(j)}$ and $P_1^{(j)}$ entirely from X'_0 and X'_1, i.e., we set $X'_0 := X'_0 \setminus P_0^{(j)}$ and $X'_1 := X'_1 \setminus P_1^{(j)}$. We also add the index j to the set $\mathcal{J}$ in this case.

When the procedure terminates, define $\mathcal{Y} := [p] \setminus \mathcal{J}$, and for $j \in \mathcal{Y}$ set $(Q_0^{(j)}, Q_1^{(j)}) := (P_0^{(j)} \cap X'_0, P_1^{(j)} \cap X'_1)$. The sets X'_i obviously satisfy properties (b) and (c). We now turn to verifying property (a). This relies on the following claim.

Claim 5.5.1. *If* $j \in [p] \setminus \mathcal{J}$*, then* $|P_0^{(j)} \cap X_0^a| \leqslant \frac{|P_0^{(j)}|}{4}$ *and* $|P_1^{(j)} \cap X_1^a| \leqslant \frac{|P_1^{(j)}|}{4}$.

Proof of Claim 5.5.1. Recall that E_1 is the relevant underlying edge set when working with the pairs $(P_0^{(j)}, P_1^{(j)})$. Also, recall that only vertices from $Y \cup X_0^a$ were removed from $P_0^{(j)}$, and only vertices from $Y \cup X_1^a \cup X_1^c$ were removed from $P_1^{(j)}$.

Since $j \notin \mathcal{J}$, the pair $(P_0^{(j)} \setminus Y, P_1^{(j)} \setminus (Y \cup X_1^c))$ is 2ε-regular of density at least $0.9d$ by Fact 2.1. Let

$$K_0 := \left\{v \in P_0^{(j)} \setminus Y \ : \ \mathrm{deg}_1(v, P_1^{(j)} \setminus (Y \cup X_1^c)) < 0.8d|P_1^{(j)} \setminus (Y \cup X_1^b)|\right\} ,$$
$$K_1 := \left\{v \in P_1^{(j)} \setminus (Y \cup X_1^c) \ : \ \mathrm{deg}_1(v, P_0^{(j)} \setminus Y) < 0.8d|P_0^{(j)} \setminus Y|\right\} .$$

By Fact 2.2, we have $|K_0| \leqslant 2\varepsilon|P_0^{(j)} \setminus Y| \leqslant 0.1d|P_0^{(j)}|$ and $|K_1| \leqslant 0.1d|P_1^{(j)}|$. In particular, we have

(5.25)
$$\begin{aligned}\mathrm{mindeg}_1(P_0^{(j)} \setminus (Y \cup K_0), P_1^{(j)} \setminus (Y \cup X_1^c \cup K_1)) &\geqslant 0.8d|P_1^{(j)} \setminus (Y \cup X_1^c)| - |K_1| \\ &\geqslant 0.8d \cdot 0.75|P_1^{(j)}| - 0.1d|P_1^{(j)}| \\ &> 0.25d|P_1^{(j)}| \ ,\end{aligned}$$

(5.26)
$$\begin{aligned}\mathrm{mindeg}_1(P_1^{(j)} \setminus (Y \cup X_1^c \cup K_1), P_0^{(j)} \setminus (Y \cup K_0)) &\geqslant 0.8d|P_0^{(j)} \setminus Y| - |K_0| \\ &\geqslant 0.8d \cdot 0.75|P_0^{(j)}| - 0.1d|P_0^{(j)}| \\ &> 0.25d|P_0^{(j)}| \ .\end{aligned}$$

Then (5.25) and (5.26) allow us to prove that $P_i^{(j)} \cap X_i^a \subseteq K_i$ for $i = 0, 1$. Indeed, assume inductively that $P_i^{(j)} \cap X_i^a \subseteq K_i$ for $i = 0, 1$ throughout the cleaning process until a certain step. Then (5.25) and (5.26) assert that no vertex outside of $P_0^{(j)} \setminus (Y \cup K_0)$ or $P_1^{(j)} \setminus (Y \cup X_1^c \cup K_1)$ can be removed because of (5.24), proving the induction step. The claim follows. □

Putting together the definition of $\mathcal{J}$ (through which one controls the size of $P_i^{(j)} \cap (Y \cup X_i^c)$) and Claim 5.5.1 (which controls the size of $P_i^{(j)} \cap X_i^a$), we get for each $j \in \mathcal{Y}$ and $i = 0, 1$
$$|Q_i^{(j)}| \geqslant \frac{|P_i^{(j)}|}{2} \geqslant \frac{\mu k}{2} \ .$$
Therefore, these pairs are 4ε-regular (cf. Fact 2.1). We get the property of $(4\varepsilon, \frac{d}{4})$-superregularity from the definition of X_i^c (cf. (5.24)). Thus, the pairs $(Q_0^{(j)}, Q_1^{(j)})$ are as required for Lemma 5.5 and satisfy its property (a).

The only thing we have to prove is that the set X_1' is nonempty. By the definition, for each $j \in \mathcal{J}$, we have either $|P_1^{(j)}| \leqslant 4(|(Y \cup X_1^c) \cap P_1^{(j)}|)$ or $|P_0^{(j)}| \leqslant 4|Y \cap P_0^{(j)}|$. We use that $|P_0^{(j)}| = |P_1^{(j)}|$ to see that

(5.27)
$$\left| \bigcup_{\mathcal{J}} P_1^{(j)} \right| \leqslant 4(|Y| + |X_1^c|) \ .$$

For $i \in \{1, \dots, r\}$ and for $v \in X_i \setminus X_i'$, write
$$\begin{aligned} f_{i+1}(v) &:= \deg_{i+1}(v, X_{i+1} \setminus X_{i+1}'(v)) \ , \\ g_i(v) &:= \deg_i(v, X_{i-1}'(v)) \ . \end{aligned}$$
where the sets $X_1'(v), X_{i-1}(v)'$, and $X_{i+1}'(v)$ above refer to the sets $x - 1'$, X_{i-1}', and X_{i+1}', respectively, at the moment[9] just before v is removed from X_i' (we do not define $f_{i+1}(v)$ for $i = r$).

Observe that for each $i \in \{2, \dots, r\}$, we have

(5.28)
$$\sum_{v \in X_i^b} g_i(v) < \delta k n \ .$$

[9]If $v \in Y$, then this moment is the zeroth step.

We set $X_r^c := \emptyset$. For a given $i \in [r-1]$ we have

$$\begin{aligned} |X_i^c| \cdot \frac{\gamma k}{2} &\leqslant \sum_{v \in X_i^c} f_{i+1}(v) \\ \text{(see Fig 3)} \quad &\leqslant \sum_{v \in X_{i+1} \setminus X'_{i+1}} g_{i+1}(v) \\ \text{(by 1., 5., (5.28))} \quad &< \delta k n + |X_{i+1}^c| \Omega k \,, \end{aligned} \tag{5.29}$$

as $X_i \setminus X'_i \subseteq X_i^b \cup X_i^c \cup Y$ for $i = 2, \dots, r$. Using (5.29), we deduce inductively that $|X_{r-j}^c| \leqslant (\frac{8\Omega}{\gamma})^j \delta n$ for $j = 1, 2, \dots, r-1$, and in particular that

$$|X_1^c| \leqslant \left(\frac{8\Omega}{\gamma}\right)^{r-1} \delta n \,. \tag{5.30}$$

As $X_1^a = \emptyset$, we obtain that

$$\begin{aligned} |X'_1| &= \left| X_1 \setminus \left(\bigcup_{j \in \mathcal{J}} P_1^{(j)} \cup \bigcup_{j \in \mathcal{Y}} \left(P_1^{(j)} \cap (Y \cup X_1^a \cup X_1^c) \right) \right) \right| \\ \text{(by (5.27))} \quad &\geqslant |X_1| - 4(|Y| + |X_1^c|) - \left| \bigcup_{j \in \mathcal{Y}} \left(P_1^{(j)} \cap X_1^a \right) \right| \\ \text{(by 1., (5.23), (5.30))} \quad &\geqslant |X_1| - \frac{\eta n}{2} - \left| \bigcup_{j \in \mathcal{Y}} (P_1^{(j)} \cap X_1^a) \right| \\ \text{(by C5.5.1)} \quad &\geqslant |X_1| - \frac{\eta n}{2} - \frac{|X_1|}{4} \\ \text{(by 2.)} \quad &> 0 \,, \end{aligned}$$

as desired. ∎

6. Obtaining a configuration. In this section we prove that the structure in the graph $G \in \mathbf{LKS}(n, k, \eta)$ guaranteed by the main results of [HKP⁺a, HKP⁺b] always leads to one of the configurations ($\diamond$**1**)–($\diamond$**10**), as promised in Lemma 4.17. We distinguish two cases. When the set $\mathbb{H}$ of vertices of huge degree (coming from a sparse decomposition of G) is incident to many edges, then one of the configurations ($\diamond$**1**)–($\diamond$**5**) must occur (cf. Lemma 6.1). Otherwise, when the edges incident to $\mathbb{H}$ can be neglected, we obtain one of the configurations ($\diamond$**6**)–($\diamond$**10**) (cf. Lemmas 6.2 and 6.3).

Lemmas 6.1, 6.2, and 6.3 are stated in the first subsection of this section, and their proofs occupy sections 6.3, 6.4, and 6.5, respectively. The proof of Lemma 4.17 is in section 6.2.

6.1. Statements of the auxiliary lemmas. The proof of the main result of this paper, Lemma 4.17, relies on Lemmas 6.1–6.3 below. For an input graph $G_{\mathrm{L}4.17}$ one of these lemmas is applied depending on the majority type of "good" edges in $G_{\mathrm{L}4.17}$. Observe that (**K1**) of [HKP⁺b, Lemma 5.4] guarantees edges between $\mathbb{H}$ and $\mathbb{XA} \cup \mathbb{XB}$, or between $\mathbb{XA}$ and $\mathbb{XA} \cup \mathbb{XB}$ either in $E(G_{\exp})$ or in $E(G_{\mathcal{D}})$. Lemma 6.1 is used if we find edges between $\mathbb{H}$ and $\mathbb{XA} \cup \mathbb{XB}$. Lemma 6.2 is used if we find edges of $E(G_{\exp})$ between $\mathbb{XA}$ and $\mathbb{XA} \cup \mathbb{XB}$. The remaining case can be reduced to the setting of Lemma 6.3. Lemma 6.3 is also used to obtain a configuration if we are in case (**K2**) of [HKP⁺b, Lemma 5.4].

LEMMA 6.1. *Suppose we are in Setting* 3.5. *Assume that*

$$e_{G_\nabla}(\mathbb{H}, \mathbb{XA} \cup \mathbb{XB}) \geqslant \frac{\eta^{13}kn}{10^{28}(\Omega^*)^3}\ . \tag{6.1}$$

Then G contains at least one of the following configurations:

- $(\diamond\mathbf{1})$,
- $(\diamond\mathbf{2})\big(\frac{\eta^{39}\Omega^{**}}{4\cdot 10^{90}(\Omega^*)^{11}}, \frac{\sqrt[4]{\Omega^{**}}}{2}, \frac{\eta^{13}\rho^2}{128\cdot 10^{30}\cdot(\Omega^*)^5}\big)$,
- $(\diamond\mathbf{3})\big(\frac{\eta^{39}\Omega^{**}}{4\cdot 10^{90}(\Omega^*)^{11}}, \frac{\sqrt[4]{\Omega^{**}}}{2}, \frac{\gamma}{2}, \frac{\eta^{13}\gamma^2}{128\cdot 10^{30}\cdot(\Omega^*)^5}\big)$,
- $(\diamond\mathbf{4})\big(\frac{\eta^{39}\Omega^{**}}{4\cdot 10^{90}(\Omega^*)^{11}}, \frac{\sqrt[4]{\Omega^{**}}}{2}, \frac{\gamma}{2}, \frac{\eta^{13}\gamma^3}{384\cdot 10^{30}(\Omega^*)^6}\big)$, *or*
- $(\diamond\mathbf{5})\big(\frac{\eta^{39}\Omega^{**}}{4\cdot 10^{90}(\Omega^*)^{11}}, \frac{\sqrt[4]{\Omega^{**}}}{2}, \frac{\eta^{13}}{128\cdot 10^{30}\cdot(\Omega^*)^3}, \frac{\eta}{2}, \frac{\eta^{13}}{128\cdot 10^{30}\cdot(\Omega^*)^4}\big)$.

LEMMA 6.2. *Suppose that we are in Settings* 3.5 *and* 3.8. *If there exist two disjoint sets $\mathbb{YA}_1, \mathbb{YA}_2 \subseteq V(G)$ such that*

$$e_{G_{\exp}}(\mathbb{YA}_1, \mathbb{YA}_2) \geqslant 2\rho kn \tag{6.2}$$

and either

$$\mathbb{YA}_1 \cup \mathbb{YA}_2 \subseteq \mathbb{XA}^{\restriction 0} \setminus (\mathbb{J} \cup \bar{V} \cup \mathbb{F}) \quad or \tag{6.3}$$

$$\mathbb{YA}_1 \subseteq \mathbb{XA}^{\restriction 0} \setminus (\mathbb{J} \cup \bar{V} \cup \mathbb{F} \cup \mathbb{J}_2 \cup \mathbb{J}_3) \quad and \quad \mathbb{YA}_2 \subseteq \mathbb{XB}^{\restriction 0} \setminus (\mathbb{J} \cup \bar{V} \cup \mathbb{F})\ , \tag{6.4}$$

then G has configuration $(\diamond\mathbf{6})(\frac{\eta^3\rho^4}{10^{14}(\Omega^*)^3}, 0, 1, 1, \frac{3\eta^3}{2\cdot 10^3}, \mathfrak{p}_2(1+\frac{\eta}{20})k)$.

LEMMA 6.3. *Suppose that we are in Settings* 3.5 *and* 3.8. *Let $\mathcal{D}_\nabla$ be as in Lemma* 3.6. *Suppose that there exists an $(\bar{\varepsilon}, \bar{d}, \beta k)$-regularized matching $\mathcal{M}$, with $V(\mathcal{M}) \subseteq \mathbb{A}_0$, $|V(\mathcal{M})| \geqslant \frac{\rho n}{\Omega^*}$, and fulfilling one of the following two properties:*

(M1) *$\mathcal{M}$ is absorbed by $\mathcal{M}_{\mathrm{good}}$, $\bar{\varepsilon} := \frac{10^5\varepsilon'}{\eta^2}$, $\bar{d} := \frac{\gamma^2}{4}$, and $\beta := \frac{\eta^2\mathfrak{c}}{8\cdot 10^3 k}$.*

(M2) *$E(\mathcal{M}) \subseteq E(\mathcal{D}_\nabla)$, $\mathcal{M}$ is absorbed by $\mathcal{D}_\nabla$, $\bar{\varepsilon} := \pi$, $\bar{d} := \frac{\gamma^3\rho}{32\Omega^*}$, and $\beta := \frac{\widehat{\alpha}\rho}{\Omega^*}$.*

Suppose further that **(cA)** *or* **(cB)** *occurs.*

(cA) *$V(\mathcal{M}) \subseteq \mathbb{XA}^{\restriction 0} \setminus (\mathbb{J} \cup \bar{V} \cup \mathbb{F})$, and we have for the set*

$$R := \mathrm{shadow}_{G_\nabla}\left((V_{\rightsquigarrow\mathbb{E}} \cap \mathbb{L}_{\eta,k}(G)) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B), \frac{2\eta^2 k}{10^5}\right)$$

one of the following:

- **(t1)** $V_1(\mathcal{M}) \subseteq \mathrm{shadow}_{G_\nabla}(V(G_{\exp}), \rho k)$,
- **(t2)** $V_1(\mathcal{M}) \subseteq V_{\rightsquigarrow\mathbb{E}}$,
- **(t3)** $V_1(\mathcal{M}) \subseteq R \setminus (\mathrm{shadow}_{G_\nabla}(V(G_{\exp}), \rho k) \cup V_{\rightsquigarrow\mathbb{E}})$, *or*
- **(t5)** $V(\mathcal{M}) \subseteq V(G_{\mathrm{reg}}) \setminus (\mathrm{shadow}_{G_\nabla}(V(G_{\exp}), \rho k) \cup V_{\rightsquigarrow\mathbb{E}} \cup R)$.

(cB) *$V_1(\mathcal{M}) \subseteq \mathbb{XA}^{\restriction 0} \setminus (\mathbb{J} \cup \mathbb{J}_2 \cup \mathbb{J}_3 \cup \bar{V} \cup \mathbb{F})$ and $V_2(\mathcal{M}) \subseteq \mathbb{XB}^{\restriction 0} \setminus (\mathbb{J} \cup \bar{V} \cup \mathbb{F})$, and we have*

- **(t1)** $V_1(\mathcal{M}) \subseteq \mathrm{shadow}_{G_\nabla}(V(G_{\exp}), \rho k)$,
- **(t2)** $V_1(\mathcal{M}) \subseteq V_{\rightsquigarrow\mathbb{E}}$, *or*
- **(t3–5)** $V_1(\mathcal{M}) \cap (\mathrm{shadow}_{G_\nabla}(V(G_{\exp}), \rho k) \cup V_{\rightsquigarrow\mathbb{E}}) = \emptyset$.

Then at least one of the following configurations occurs:

- $(\diamond\mathbf{6})\big(\frac{\eta^3\rho^4}{10^{12}(\Omega^*)^4}, 4\pi, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4}, \frac{3\eta^3}{2000}, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)$,
- $(\diamond\mathbf{7})\big(\frac{\eta^3\gamma^3\rho}{10^{12}(\Omega^*)^4}, \frac{\eta\gamma}{400}, 4\pi, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4}, \frac{3\eta^3}{2000}, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)$,

- $(\diamond\mathbf{8})\big(\frac{\eta^4\gamma^4\rho}{10^{15}(\Omega^*)^5}, \frac{\eta\gamma}{400}, \frac{400\varepsilon}{\eta}, 4\pi, \frac{d}{2}, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta\pi\mathfrak{c}}{200k}, \frac{\eta^2\nu}{2\cdot 10^4}, \mathfrak{p}_1(1+\frac{\eta}{20})k, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)$,
- $(\diamond\mathbf{9})\big(\frac{\rho\eta^8}{10^{27}(\Omega^*)^3}, \frac{2\eta^3}{10^3}, \mathfrak{p}_1(1+\frac{\eta}{40})k, \mathfrak{p}_2(1+\frac{\eta}{20})k, \frac{400\varepsilon}{\eta}, \frac{d}{2}, \frac{\eta\pi\mathfrak{c}}{200k}, 4\pi, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4}\big)$,
- $(\diamond\mathbf{10})\big(\varepsilon, \frac{\gamma^2 d}{2}, \pi\sqrt{\varepsilon'}\nu k, \frac{(\Omega^*)^2 k}{\gamma^2}, \frac{\eta}{40}\big)$.

6.2. Proof of Lemma 4.17. Throughout this section (and including subordinate lemmas), we assume that we have the setting of Lemma 4.17. In particular, we shall assume Settings 3.5 and 3.8.

We distinguish different types of edges captured in cases (**K1**) and (**K2**). If in case (**K1**) many of the captured edges from $\mathbb{XA}$ to $\mathbb{XA}\cup\mathbb{XB}$ are incident to $\mathbb{H}$, we will get one of the configurations $(\diamond\mathbf{1})$–$(\diamond\mathbf{5})$ by employing Lemma 6.1. Otherwise, there must be many edges from $\mathbb{XA}$ to $\mathbb{XA}\cup\mathbb{XB}$ in the graph G_{exp} or in $G_{\mathcal{D}}$. Lemma 6.2 shows that the former case leads to configuration $(\diamond\mathbf{6})$. We will reduce the latter case to the situation in Lemma 6.3 which gives one of the configurations $(\diamond\mathbf{6})$–$(\diamond\mathbf{10})$.

We use Lemma 6.3 to give one of the configurations $(\diamond\mathbf{6})$–$(\diamond\mathbf{10})$ also in the case (**K2**).[10]

Let us now turn to the details of the proof. If $e_{G_\nabla}(\mathbb{H},\mathbb{XA}\cup\mathbb{XB}) \geqslant \frac{\eta^{13}kn}{10^{28}(\Omega^*)^3}$, then we use Lemma 6.1 to obtain one of the configurations $(\diamond\mathbf{1})$–$(\diamond\mathbf{5})$, with the parameters as in the statement of Lemma 4.17.

Recall that every edge of G incident to $\mathbb{H}$ is captured. Thus, in the remainder of the proof we assume that

$$e_G(\mathbb{H},\mathbb{XA}\cup\mathbb{XB}) = e_{G_\nabla}(\mathbb{H},\mathbb{XA}\cup\mathbb{XB}) < \frac{\eta^{13}kn}{10^{28}(\Omega^*)^3}\ . \tag{6.5}$$

We now bound the size of the set $\mathbb{J}$. By Setting 3.5(9) we have that

$$|E(G)\setminus E(G_\nabla)| \leqslant 2\rho kn.$$

We shall therefore use Lemma 3.10 with $\beta_{\mathrm{L3.10}} = 2\rho$. This choice of $\beta_{\mathrm{L3.10}}$ is consistent with (3.26); indeed, by (3.4) we have that $\eta \gg \rho \gg \gamma$, and thus $\rho \gg \eta^2\sqrt{\gamma}$.[11] From Lemma 3.10 we get $|L_\#| \leqslant \frac{40\rho n}{\eta}$, $|\mathbb{XA}\setminus\mathbb{YA}| \leqslant \frac{1200\rho n}{\eta^2}$, and $|(\mathbb{XA}\cup\mathbb{XB})\setminus\mathbb{YB}| \leqslant \frac{1200\rho n}{\eta^2}$. Further, using (6.5), Lemma 3.10 also gives that $|V_{\rightsquigarrow\mathbb{H}}| \leqslant \frac{\eta^{12}n}{10^{26}(\Omega^*)^3}$. It follows from Setting 3.5(8) that $|\mathbb{J}_\mathbb{E}| \leqslant \gamma n$. Lastly, by Setting 3.5(7) we have $|\mathbb{J}_1| \leqslant 2\gamma n$. Thus,

$$\begin{aligned}|\mathbb{J}| &\leqslant |\mathbb{XA}\setminus\mathbb{YA}| + |(\mathbb{XA}\cup\mathbb{XB})\setminus\mathbb{YB}| + |V_{\rightsquigarrow\mathbb{H}}| + |L_\#| + |\mathbb{J}_1| \\ &\quad + \left|\mathrm{shadow}_{G_{\mathcal{D}}\cup G_\nabla}\left(V_{\rightsquigarrow\mathbb{H}}\cup L_\#\cup\mathbb{J}_\mathbb{E}\cup\mathbb{J}_1, \frac{\eta^2 k}{10^5}\right)\right| \\ &\overset{(3.4)}{\leqslant} \frac{2\eta^{10}n}{10^{21}(\Omega^*)^2}\ ,\end{aligned} \tag{6.6}$$

where we used Fact 3.1 to bound the size of the shadows (to this end recall that by property 1 of Definition 2.11, the graph $G_{\mathcal{D}}\cup G_\nabla$ indeed has maximum degree at most $\Omega^* k$).

Let us first turn our attention to case (**K1**). By Definition 3.7 we have $\mathbb{H}\cap\mathbb{A}_0 = \emptyset$.

[10] Actually, our proof of Lemma 6.3 implies that one does not get configuration $(\diamond\mathbf{9})$ in case (**K2**), but this fact is never needed.

[11] Recall that the choice of constants in (3.4) proceeds from left to right.

Therefore,

$$
\begin{aligned}
e_{G_\nabla}\big(\mathbb{XA}^{\restriction 0}\setminus\mathbb{J},(\mathbb{XA}\cup\mathbb{XB})^{\restriction 0}\setminus\mathbb{J}\big) &= e_{G_\nabla}\big((\mathbb{XA}\setminus(\mathbb{H}\cup\mathbb{J}))^{\restriction 0},(\mathbb{XA}\setminus(\mathbb{H}\cup\mathbb{J}))^{\restriction 0}\cup(\mathbb{XB}\setminus\mathbb{J})^{\restriction 0}\big)\\
\text{(by D3.7(7))}\quad &\geqslant \mathfrak{p}_0^2\cdot e_{G_\nabla}\big(\mathbb{XA}\setminus(\mathbb{H}\cup\mathbb{J}),(\mathbb{XA}\cup\mathbb{XB})\setminus(\mathbb{H}\cup\mathbb{J})\big)-k^{0.6}n^{0.6}\\
\text{(by (3.18))}\quad &\geqslant \frac{\eta^2}{10^4}\big(e_{G_\nabla}(\mathbb{XA},\mathbb{XA}\cup\mathbb{XB})-2e_{G_\nabla}(\mathbb{H},\mathbb{XA}\cup\mathbb{XB})-2|\mathbb{J}|\Omega^*k\big)-k^{0.6}n^{0.6}\\
\text{(by (K1), (6.5), (6.6))}\quad &\geqslant \frac{\eta^2}{10^4}\Big(\frac{\eta kn}{4}-\frac{2\eta^{13}kn}{10^{28}(\Omega^*)^3}-\frac{4\eta^{10}kn}{10^{21}\Omega^*}\Big)-k^{0.6}n^{0.6}\\
&> \frac{\eta^3 kn}{10^5}\ . \qquad (6.7)
\end{aligned}
$$

We consider the following two complementary cases:

(**wA**) $e_{G_\nabla}((\mathbb{XA}\setminus\mathbb{J})^{\restriction 0})\geqslant 40\rho kn$.

(**wB**) $e_{G_\nabla}((\mathbb{XA}\setminus\mathbb{J})^{\restriction 0})< 40\rho kn$.

Note that $\mathbb{XA}\setminus\mathbb{J}\subseteq\mathbb{YA}$ and $(\mathbb{XA}\cup\mathbb{XB})\setminus\mathbb{J}\subseteq\mathbb{YB}$. We shall now define in each of the cases (**wA**) and (**wB**) certain sets $\mathbb{YA}_1,\mathbb{YA}_2$. The way these sets are defined will guarantee a lower bound on the number of edges between them. Although the definition of these sets is different for the cases (**wA**) and (**wB**), for ease of notation they receive the same names.

In case (**wA**) a standard argument (take a maximal cut) gives disjoint sets $\mathbb{YA}_1,\mathbb{YA}_2\subseteq(\mathbb{XA}\setminus(\mathbb{J}\cup\bar V\cup\mathbb{F}))^{\restriction 0}\subseteq\mathbb{YA}$ with

$$
\begin{aligned}
e_{G_\nabla}(\mathbb{YA}_1,\mathbb{YA}_2) &\geqslant \frac{1}{2}(e_{G_\nabla}(\mathbb{XA}\setminus\mathbb{J})^{\restriction 0}-|\bar V\cup\mathbb{F}|\cdot\Omega^*k)\\
\text{(by D3.7(1) and (3.20))}\quad &\geqslant \frac{1}{2}(40\rho kn-2\varepsilon\Omega^*kn)\\
&> 19\rho kn\ . \qquad (6.8)
\end{aligned}
$$

Let us now define $\mathbb{YA}_1,\mathbb{YA}_2$ for case (**wB**). Setting 3.5(6) implies that

$$
|\mathbb{J}_2|\leqslant\sqrt{\gamma}n\ . \qquad (6.9)
$$

Also, by Definition 3.7(7) we have

$$
\begin{aligned}
e_{G_\nabla}(\mathbb{XA}) &\leqslant \frac{1}{\mathfrak{p}_0^2}\big(e_{G_\nabla}((\mathbb{XA}\setminus\mathbb{J})^{\restriction 0})+k^{0.6}n^{0.6}\big)+e_{G_\nabla}(\mathbb{H},\mathbb{XA})+|\mathbb{J}|\Omega^*k\\
\text{(by (3.18), (wB), (6.5), and (6.6))}\quad &\leqslant \frac{10^4}{\eta^2}\cdot\big(40\rho kn+k^{0.6}n^{0.6}\big)+\frac{\eta^{13}}{10^{28}(\Omega^*)^3}kn+\frac{\eta^{10}}{10^{20}\Omega^*}kn\\
\text{(by (3.4))}\quad &< \frac{\eta^8}{10^{15}\Omega^*}kn\ .
\end{aligned}
$$

Consequently,

$$
|\mathbb{J}_3|\cdot\frac{\eta^3k}{10^3}\leqslant e_{G_\nabla}(\mathbb{J}_3,\mathbb{XA})\leqslant 2\cdot\frac{\eta^8}{10^{15}\Omega^*}kn,
$$

and thus,

$$
|\mathbb{J}_3|\leqslant 2\cdot\frac{\eta^5}{10^{12}\Omega^*}n\ . \qquad (6.10)
$$

Set $\mathbb{YA}_1 := (\mathbb{XA} \setminus (\mathbb{J} \cup \mathbb{J}_2 \cup \mathbb{J}_3 \cup \bar{V} \cup \mathbb{F}))^{\restriction 0} \subseteq \mathbb{YA}$ and $\mathbb{YA}_2 := (\mathbb{XB} \setminus (\mathbb{J} \cup \bar{V} \cup \mathbb{F}))^{\restriction 0} \subseteq \mathbb{YB}$. Then the sets $\mathbb{YA}_1$ and $\mathbb{YA}_2$ are disjoint and we have

$$\begin{aligned} e_{G_\nabla}(\mathbb{YA}_1, \mathbb{YA}_2) &\geqslant e_{G_\nabla}\left((\mathbb{XA} \setminus \mathbb{J})^{\restriction 0}, ((\mathbb{XA} \cup \mathbb{XB}) \setminus \mathbb{J})^{\restriction 0}\right) \\ &\quad - 2e_{G_\nabla}((\mathbb{XA} \setminus \mathbb{J})^{\restriction 0}) - (|\mathbb{J}_2| + |\mathbb{J}_3| + 2|\bar{V}| + 2|\mathbb{F}|) \cdot \Omega^* k \\ \text{(by (6.7), (}\mathbf{wB}\text{), (6.9), (6.10), D3.7(1), (3.20))} \quad &\geqslant \frac{\eta^3 kn}{10^5} - 80\rho kn - \sqrt{\gamma}\Omega^* kn - \frac{2\eta^5}{10^{12}} kn - 4\varepsilon\Omega^* kn \\ &\overset{(3.4)}{\geqslant} 19\rho kn \ . \end{aligned} \tag{6.11}$$

We have thus defined $\mathbb{YA}_1, \mathbb{YA}_2$ for both cases (**wA**) and (**wB**).

Observe first that if $e_{G_{\exp}}(\mathbb{YA}_1, \mathbb{YA}_2) \geqslant 2\rho kn$, then we may apply Lemma 6.2 to obtain configuration $(\diamond\mathbf{6})(\frac{\eta^3\rho^4}{10^{14}(\Omega^*)^3}, 0, 1, 1, \frac{3\eta^3}{2\cdot 10^3}, \mathfrak{p}_2(1+\frac{\eta}{20})k)$. Hence, from now on, let us assume that $e_{G_{\exp}}(\mathbb{YA}_1, \mathbb{YA}_2) > 2\rho kn$. Then by (6.8) and (6.11) we have that

$$e_{G_\mathcal{D}}(\mathbb{YA}_1, \mathbb{YA}_2) \geqslant 17\rho kn \ .$$

We fix a family $\mathcal{D}_\nabla$ as in Lemma 3.6. In particular, we have

$$e_{\mathcal{D}_\nabla}(\mathbb{YA}_1, \mathbb{YA}_2) \geqslant 16\rho kn \ , \tag{6.12}$$

$$\mathrm{maxdeg}(\mathcal{D}_\nabla) \leqslant \mathrm{maxdeg}(\mathcal{D}) \overset{\text{D2.11(1)}}{\leqslant} \Omega^* k \ . \tag{6.13}$$

Let $R := \mathrm{shadow}_{G_\nabla}((V_{\leadsto\mathbb{E}} \cap \mathbb{L}_{\eta,k}(G)) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B), \frac{2\eta^2 k}{10^5})$. For $i = 1, 2$ define

$$\begin{aligned} \mathbb{Y}_i^{(1)} &:= \mathrm{shadow}_G(V(G_{\exp}), \rho k) \cap \mathbb{YA}_i \ , \\ \mathbb{Y}_i^{(2)} &:= (V_{\leadsto\mathbb{E}} \cap \mathbb{YA}_i) \setminus \mathbb{Y}_i^{(1)} \ , \\ \mathbb{Y}_i^{(3)} &:= (R \cap \mathbb{YA}_i) \setminus (\mathbb{Y}_i^{(1)} \cup \mathbb{Y}_i^{(2)}) \ , \\ \mathbb{Y}_i^{(4)} &:= (\mathbb{E} \cap \mathbb{YA}_i) \setminus (\mathbb{Y}_i^{(1)} \cup \mathbb{Y}_i^{(2)} \cup \mathbb{Y}_i^{(3)}) \ , \\ \mathbb{Y}_i^{(5)} &:= \mathbb{YA}_i \setminus (\mathbb{Y}_i^{(1)} \cup \cdots \cup \mathbb{Y}_i^{(4)}) \ . \end{aligned} \tag{6.14}$$

Clearly, the sets $\mathbb{Y}_i^{(j)}$ partition $\mathbb{YA}_i$ for $i = 1, 2$.

We now present two lemmas (one for case (**wA**) and one for case (**wB**)) which help to distinguish several subcases based on the majority type of edges we find between $\mathbb{YA}_1$ and $\mathbb{YA}_2$. The first of the two lemmas follows by a simple counting argument from (6.12).

LEMMA 6.4. *In case* (**wB**), *we have one of the following:*

(**t1**) $e_{\mathcal{D}_\nabla}\left(\mathbb{Y}_1^{(1)}, \mathbb{YA}_2\right) \geqslant 2\rho kn$,
(**t2**) $e_{\mathcal{D}_\nabla}\left(\mathbb{Y}_1^{(2)}, \mathbb{YA}_2\right) \geqslant 2\rho kn$,
(**t3**) $e_{\mathcal{D}_\nabla}\left(\mathbb{Y}_1^{(3)}, \mathbb{YA}_2\right) \geqslant 2\rho kn$,
(**t4**) $e_{\mathcal{D}_\nabla}\left(\mathbb{Y}_1^{(4)}, \mathbb{YA}_2\right) \geqslant 2\rho kn$, *or*
(**t5**) $e_{\mathcal{D}_\nabla}\left(\mathbb{Y}_1^{(5)}, \mathbb{YA}_2\right) \geqslant 2\rho kn$.

Our second lemma is a bit more involved.

LEMMA 6.5. *In case* (**wA**), *we have one of the following:*

(**t1**) $e_{\mathcal{D}_\nabla}(\mathbb{Y}_1^{(1)}, \mathbb{YA}_2) + e_{\mathcal{D}_\nabla}(\mathbb{YA}_1, \mathbb{Y}_2^{(1)}) \geqslant 4\rho kn$,
(**t2**) $e_{\mathcal{D}_\nabla}\left(\mathbb{Y}_1^{(2)}, \mathbb{YA}_2 \setminus \mathbb{Y}_2^{(1)}\right) + e_{\mathcal{D}_\nabla}\left(\mathbb{YA}_1 \setminus \mathbb{Y}_1^{(1)}, \mathbb{Y}_2^{(2)}\right) \geqslant 4\rho kn$,

(**t3**) $e_{\mathcal{D}_\nabla}\big(\mathbb{Y}_1^{(3)}, \mathbb{YA}_2 \setminus (\mathbb{Y}_2^{(1)} \cup \mathbb{Y}_2^{(2)})\big) + e_{\mathcal{D}_\nabla}\big(\mathbb{YA}_1 \setminus (\mathbb{Y}_1^{(1)} \cup \mathbb{Y}_1^{(2)}), \mathbb{Y}_2^{(3)}\big) \geqslant 4\rho kn$, *or*

(**t5**) $e_{\mathcal{D}_\nabla}\big(\mathbb{Y}_1^{(5)}, \mathbb{Y}_2^{(5)}\big) \geqslant 2\rho kn$.

Proof. By (6.12), we need only establish that

$$e_{\mathcal{D}_\nabla}\left(\mathbb{Y}_1^{(4)}, \mathbb{YA}_2 \setminus (\mathbb{Y}_2^{(1)} \cup \mathbb{Y}_2^{(2)} \cup \mathbb{Y}_2^{(3)})\right) + e_{\mathcal{D}_\nabla}\left(\mathbb{YA}_1 \setminus (\mathbb{Y}_1^{(1)} \cup \mathbb{Y}_1^{(2)} \cup \mathbb{Y}_1^{(3)}), \mathbb{Y}_2^{(4)}\right) < \rho kn\ .$$

For this, note that $\mathbb{Y}_1^{(4)} \subseteq \mathbb{E}$ and that $\mathbb{YA}_2 \setminus (\mathbb{Y}_2^{(1)} \cup \mathbb{Y}_2^{(2)} \cup \mathbb{Y}_2^{(3)})$ is disjoint from $V_{\rightsquigarrow\mathbb{E}}$. Thus we have $e_{\mathcal{D}_\nabla}(\mathbb{Y}_1^{(4)}, \mathbb{YA}_2 \setminus (\mathbb{Y}_2^{(1)} \cup \mathbb{Y}_2^{(2)} \cup \mathbb{Y}_2^{(3)})) < \frac{\rho kn}{100\Omega^*}$. We can bound the other summand using a symmetric argument. ∎

We can now provide a crucial step for finishing case (**K1**).

Lemma 6.6. *Let G^* be the spanning subgraph of $G_{\mathcal{D}}$ formed by the edges of $\mathcal{D}_\nabla$. If there are two disjoint sets Z_1 and Z_2 with $e_{G^*}(Z_1, Z_2) \geqslant 2\rho kn$, then there exists a $(\pi, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\widehat{\alpha}\rho k}{\Omega^*})$-regularized matching $\mathcal{N}$ in G^*, with $V_i(\mathcal{N}) \subseteq Z_i$ $(i = 1, 2)$ and $|V(\mathcal{N})| \geqslant \frac{\rho n}{\Omega^*}$.*

Proof. By (6.13), the maximum degree of G^* is bounded by $\Omega^* k$. Therefore, we have $|Z_1| \geqslant \frac{2\rho n}{\Omega^*} \geqslant \frac{2\rho k}{\Omega^*}$. Thus,

$$(G^*, \mathcal{D}_\nabla, G^*[Z_1, Z_2], \{Z_1\}) \in \mathcal{G}\left(v(G_{\mathcal{D}}), k, \Omega^*, \frac{\gamma^3}{4}, \frac{\rho}{\Omega^*}, 2\rho\right)\ ,$$

where the class of the right-hand side was defined in Definition 2.14. Lemma 2.15 (which applies with these parameters by the choice of $\widehat{\alpha}$ and k_0 by (3.4)) immediately gives the desired output. ∎

We use Lemma 6.6 with Z_1, Z_2 being the pair of sets containing many edges as in the cases (**t1**)–(**t3**) and (**t5**) of Lemma 6.5[12] and (**t1**)–(**t5**) of Lemma 6.4. Lemma 6.6 outputs a regularized matching $\mathcal{M}_{\mathrm{L6.3}} := \mathcal{N}_{\mathrm{L6.6}}$. This matching is a basis of the input for Lemma 6.3(**M2**) (subcase (**t1**)–(**t3**), (**t5**), or (**t3–5**)). Thus, we get one of the configurations ($\diamond$**6**)–($\diamond$**10**) as in the statement of the lemma. This finishes the proof for case (**K1**).

Let us now turn our attention to case (**K2**). For every pair $(X, Y) \in \mathcal{M}_{\mathrm{good}}$, let $X' \subseteq X^{\upharpoonright 0} \setminus (\mathbb{J} \cup \bar{V} \cup \mathbb{F})$ and $Y' \subseteq Y^{\upharpoonright 0} \setminus (\mathbb{J} \cup \bar{V} \cup \mathbb{F})$ be maximal with $|X'| = |Y'|$. Define $\mathcal{N} := \{(X', Y') : (X, Y) \in \mathcal{M}_{\mathrm{good}}\ ,\ |X'| \geqslant \frac{\eta^2\mathfrak{c}}{2\cdot 10^3}\}$. By Lemma 3.9, and using (3.4) and (3.18), we know that

$$|V(\mathcal{M}_{\mathrm{good}}^{\upharpoonright 0})| \geqslant \frac{\eta^2 n}{400}\ .$$

Therefore, we have

$$\begin{aligned}
|V(\mathcal{N})| &\geqslant |V(\mathcal{M}_{\mathrm{good}}^{\upharpoonright 0})| - 2|\mathbb{J} \cup \bar{V} \cup \mathbb{F}| - 2\frac{\eta^2 n}{2\cdot 10^3} \\
\text{(by (}\mathbf{K2}\text{), (6.6), D3.7(1), (3.20))} \quad &\geqslant \frac{\eta^2 n}{400} - \frac{4\cdot\eta^{10} n}{10^{21}(\Omega^*)^2} - 4\varepsilon n - \frac{\eta^2 n}{10^3} \\
&> \frac{\eta^2 n}{1000}\ .
\end{aligned} \tag{6.15}$$

By Fact 2.1, $\mathcal{N}$ is a $(\frac{4\cdot 10^3\varepsilon'}{\eta^2}, \frac{\gamma^2}{2}, \frac{\eta^2\mathfrak{c}}{2\cdot 10^3})$-regularized matching.

[12] The quantities in Lemma 6.5 have two summands. We take the sets Z_1,Z_2 as those appearing in the majority summand.

We use the definitions of the sets $\mathbb{Y}_i^{(1)}, \dots, \mathbb{Y}_i^{(5)}$ as given in (6.14) with $\mathbb{YA}_i := V_i(\mathcal{N})$ $(i = 1, 2)$. As $V(\mathcal{N}) \subseteq V(G_{\mathrm{reg}})$, we have that $\mathbb{Y}_i^{(4)} = \emptyset$ $(i = 1, 2)$. A set $X \in \mathcal{V}_i(\mathcal{N})$ is said to be of *Type* 1 if $|X \cap \mathbb{Y}_i^{(1)}| \geqslant \frac{1}{4}|X|$. Analogously, we define elements of $\mathcal{V}(\mathcal{N})$ of *Type* 2, *Type* 3, and *Type* 5.

By (6.15) and as $V(\mathcal{M}_{\mathrm{good}}) \subseteq \mathbb{XA}$, we are in subcase (**wA**). For each $(X_1, X_2) \in \mathcal{N}$ with at least one $X_i \in \{X_1, X_2\}$ being of Type 1, set $X_i' := X_i \cap \mathbb{Y}_i^{(1)}$ and take an arbitrary set $X_{3-i}' \subseteq X_{3-i}$ of size $|X_i'|$. Note that by Fact 2.1 (X_i', X_{3-i}') forms a $\frac{10^5\varepsilon'}{\eta^2}$-regular pair of density at least $\gamma^2/4$. We let $\mathcal{N}_1$ be the regularized matching consisting of all pairs (X_i', X_{3-i}') obtained in this way.[13]

Likewise, we construct $\mathcal{N}_2$, $\mathcal{N}_3$, and $\mathcal{N}_5$ using the features of Types 2, 3, and 5. Observe that the matchings $\mathcal{N}_i$ may intersect.

Because of (6.15) and since we included at least one quarter of each $\mathcal{N}$-edge into one of $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$, and $\mathcal{N}_5$, one of the regularized matchings $\mathcal{N}_i$ satisfies $|V(\mathcal{N}_i)| \geqslant \frac{\eta^2 n}{16\cdot 1000} \geqslant \frac{\rho}{\Omega^*}n$. So, $\mathcal{N}_i$ serves as a matching $\mathcal{M}_{\mathrm{L6.3}}$ for Lemma 6.3(**M1**). Thus, we get one of the configurations ($\diamond$**6**)–($\diamond$**10**) as in the statement of the lemma. This finishes case (**K2**).

6.3. Proof of Lemma 6.1. Set $\tilde{\eta} := \frac{\eta^{13}}{10^{28}(\Omega^*)^3}$. Define $\mathrm{N}^{\uparrow} := \{v \in V(G) : \deg_{G_\nabla}(v, \mathbb{H}) \geqslant k\}$ and $\mathrm{N}^{\downarrow} := \mathrm{N}_{G_\nabla}(\mathbb{H}) \setminus \mathrm{N}^{\uparrow}$. Recall that by the definition of the class **LKSsmall**(n, k, η), the set $\mathbb{H}$ is independent, and thus the sets $\mathrm{N}^{\uparrow}$ and $\mathrm{N}^{\downarrow}$ are disjoint from $\mathbb{H}$. Also, using the same definition, we have

$$\mathrm{N}_{G_\nabla}(\mathbb{H}) \subseteq \mathbb{L}_{\eta,k}(G) \setminus \mathbb{H}, \quad \text{and thus} \tag{6.16}$$

$$e_{G_\nabla}(\mathbb{H}, B) = e_{G_\nabla}(\mathbb{H}, B \cap \mathbb{L}_{\eta,k}(G)) \text{ for any } B \subseteq V(G). \tag{6.17}$$

We shall distinguish two cases.

Case A: $e_{G_\nabla}(\mathbb{H}, \mathrm{N}^{\uparrow}) \geqslant e_{G_\nabla}(\mathbb{H}, \mathbb{XA} \cup \mathbb{XB})/8$. Let us focus on the bipartite subgraph H' of G_∇ induced by the sets $\mathbb{H}$ and $\mathrm{N}^{\uparrow}$. Obviously, the average degree of the vertices of $\mathrm{N}^{\uparrow}$ in H' is at least k.

First, suppose that $|\mathbb{H}| \leqslant |\mathrm{N}^{\uparrow}|$. Then, the average degree of $\mathbb{H}$ in H' is at least k, and hence, the average degree of H' is at least k. Thus, there exists a bipartite subgraph $H \subseteq H'$ with $\mathrm{mindeg}(H) \geqslant k/2$. Furthermore, $\mathrm{mindeg}_{G_\nabla}(V(H)) \geqslant k$. We conclude that we are in configuration ($\diamond$**1**).

Now, suppose $|\mathbb{H}| > |\mathrm{N}^{\uparrow}|$. Using the bounds given by Case A, and using (6.1), we get

$$|\mathrm{N}^{\uparrow}| \geqslant \frac{e_{G_\nabla}(\mathbb{H}, \mathrm{N}^{\uparrow})}{\Omega^* k} \geqslant \frac{\tilde{\eta} k n}{8\Omega^* k} = \frac{\tilde{\eta} n}{8\Omega^*}\ .$$

Therefore, we have

$$e(G) \geqslant \sum_{v \in \mathbb{H}} \deg_{G_\nabla}(v) \geqslant |\mathbb{H}|\Omega^{**}k > |\mathrm{N}^{\uparrow}|\Omega^{**}k \geqslant \frac{\tilde{\eta} n}{8\Omega^*}\Omega^{**}k \overset{(3.4)}{\geqslant} kn\ ,$$

a contradiction to property 3 of Definition 2.4.

Case B: $e_{G_\nabla}(\mathbb{H}, \mathrm{N}^{\uparrow}) < e_{G_\nabla}(\mathbb{H}, \mathbb{XA} \cup \mathbb{XB})/8$. Consequently, we get

$$e_{G_\nabla}(\mathbb{H}, (\mathbb{XA} \cup \mathbb{XB}) \setminus \mathrm{N}^{\uparrow}) \geqslant \frac{7}{8} e_{G_\nabla}(\mathbb{H}, \mathbb{XA} \cup \mathbb{XB}) \overset{(6.1)}{\geqslant} \frac{7}{8}\tilde{\eta} kn\ . \tag{6.18}$$

[13]Note that we are thus changing the orientation of some subpairs.

We now apply Lemma 5.1 to G_∇ with input sets $P_{\mathrm{L5.1}} := \mathbb{H}$, $Q_{\mathrm{L5.1}} := \mathbb{L}_{\eta,k}(G)\setminus\mathbb{H}$, $Y_{\mathrm{L5.1}} := \mathbb{L}_{\eta,k}(G) \setminus \mathbb{L}_{\frac{9}{10}\eta,k}(G_\nabla)$, and parameters $\psi_{\mathrm{L5.1}} := \tilde{\eta}/100$, $\Gamma_{\mathrm{L5.1}} := \Omega^*$, $\Omega_{\mathrm{L5.1}} := \Omega^{**}$, and $\Omega'_{\mathrm{L5.1}} := \tilde{\eta}^3\Omega^{**}/(4\cdot 10^6(\Omega^*)^2)$. Assumption (5.2) of the lemma follows from (6.16), and assumption (5.1) holds by the choice of $\Omega'_{\mathrm{L5.1}}$. The lemma yields three sets $L'' := Q''_{\mathrm{L5.1}}$, $L' := Q'_{\mathrm{L5.1}}$, and $\mathbb{H}' := P'_{\mathrm{L5.1}}$, and it is easy to check that they witness preconfiguration $(\clubsuit)(\frac{\tilde{\eta}^3\Omega^{**}}{4\cdot 10^6(\Omega^*)^2})$.

Recall that $e(G) \leqslant kn$. Since by the definition of $Y_{\mathrm{L5.1}}$, we have $|Y_{\mathrm{L5.1}}| \leqslant \frac{40\rho}{\eta}n$, we obtain from Lemma 5.1(d) that

$$\begin{aligned}
e_{G_\nabla}(\mathbb{H}, \mathbb{L}_{\eta,k}(G)) - e_{G_\nabla}(\mathbb{H}', L'') &\leqslant \frac{\tilde{\eta}}{100} e_{G_\nabla}(\mathbb{H}, \mathbb{L}_{\eta,k}(G)) + |Y_{\mathrm{L5.1}}|\Omega^* k \\
&\leqslant \frac{\tilde{\eta}}{100}kn + \frac{40\rho n}{\eta}\cdot\Omega^* k \\
&\overset{(3.4)}{\leqslant} \frac{\tilde{\eta}}{2}kn\ . \qquad (6.19)
\end{aligned}$$

So,

$$\begin{aligned}
e_{G_\nabla}\big(\mathbb{H}', (L'' \cap (\mathbb{XA}\cup\mathbb{XB}))\setminus \mathrm{N}^{\uparrow}\big) &\geqslant e_{G_\nabla}\big(\mathbb{H}, (\mathbb{L}_{\eta,k}(G) \cap (\mathbb{XA}\cup\mathbb{XB}))\setminus \mathrm{N}^{\uparrow}\big) \\
&\quad - \big(e_{G_\nabla}(\mathbb{H}, \mathbb{L}_{\eta,k}(G)) - e_{G_\nabla}(\mathbb{H}', L'')\big) \\
&= e_{G_\nabla}(\mathbb{H}, (\mathbb{XA}\cup\mathbb{XB})\setminus \mathrm{N}^{\uparrow}) \\
&\quad - \big(e_{G_\nabla}(\mathbb{H}, \mathbb{L}_{\eta,k}(G)) - e_{G_\nabla}(\mathbb{H}', L'')\big) \\
&\overset{(6.19)}{\geqslant} e_{G_\nabla}(\mathbb{H}, (\mathbb{XA}\cup\mathbb{XB})\setminus \mathrm{N}^{\uparrow}) - \frac{\tilde{\eta}}{2}kn \\
&\overset{(6.18)}{\geqslant} \frac{3}{8}\tilde{\eta}kn\ . \qquad (6.20)
\end{aligned}$$

We define

$$\mathbb{H}^* := \left\{ v \in \mathbb{H}' \ :\ \deg_{G_\nabla}(v, L'' \cap (\mathbb{XA}\cup\mathbb{XB}) \cap \mathrm{N}^{\downarrow}) \geqslant \sqrt{\Omega^{**}}k \right\}\ .$$

Using that $e(G) \leqslant kn$, we shall prove the following.

Lemma 6.7. *We have* $e_{G_\nabla}(\mathbb{H}^*, L'' \cap (\mathbb{XA}\cup\mathbb{XB}) \cap \mathrm{N}^{\downarrow}) \geqslant \frac{1}{8}\tilde{\eta}kn$.

Proof. Suppose otherwise. Then by (6.20), we obtain that

$$e_{G_\nabla}(\mathbb{H}'\setminus\mathbb{H}^*, L'' \cap (\mathbb{XA}\cup\mathbb{XB}) \cap \mathrm{N}^{\downarrow}) \geqslant \frac{1}{4}\tilde{\eta}kn\ .$$

On the other hand, by the definition of $\mathbb{H}^*$,

$$|\mathbb{H}'\setminus\mathbb{H}^*|\sqrt{\Omega^{**}}k \geqslant e_{G_\nabla}(\mathbb{H}'\setminus\mathbb{H}^*, L'' \cap (\mathbb{XA}\cup\mathbb{XB}) \cap \mathrm{N}^{\downarrow})\ .$$

Consequently, we have

$$|\mathbb{H}'\setminus\mathbb{H}^*| \geqslant \frac{\tilde{\eta}kn}{4\sqrt{\Omega^{**}}k} = \frac{\tilde{\eta}n}{4\sqrt{\Omega^{**}}}\ .$$

Thus, as $\mathbb{H}$ is independent,

$$e(G) \geqslant \sum_{v\in\mathbb{H}} \deg_{G_\nabla}(v) \geqslant |\mathbb{H}|\Omega^{**}k \geqslant |\mathbb{H}'\setminus\mathbb{H}^*|\Omega^{**}k \geqslant \frac{\tilde{\eta}}{4}\sqrt{\Omega^{**}}kn \overset{(3.4)}{>} kn\ ,$$

a contradiction. ∎

Let us define $O := \mathrm{shadow}_{G_\nabla}(\mathbb{E}, \gamma k)$. Next, we define

$$N_1 := V(G_{\mathrm{exp}}) \cap L'' \cap (\mathbb{XA} \cup \mathbb{XB}) \cap \mathrm{N}^{\downarrow} ,$$
$$N_2 := \mathbb{E} \cap L'' \cap (\mathbb{XA} \cup \mathbb{XB}) \cap \mathrm{N}^{\downarrow} ,$$
$$N_3 := O \cap L'' \cap (\mathbb{XA} \cup \mathbb{XB}) \cap \mathrm{N}^{\downarrow} ,$$
$$N_4 := (L'' \cap (\mathbb{XA} \cup \mathbb{XB}) \cap \mathrm{N}^{\downarrow}) \setminus (N_1 \cup N_2 \cup N_3) .$$

Observe that

$$O \cap N_4 = \emptyset . \tag{6.21}$$

Further, for $i = 1, \ldots, 4$ define

$$C_i := \left\{ v \in \mathbb{H}^* \,:\, \deg_{G_\nabla}(v, N_i) \geqslant \deg_{G_\nabla}(v, L'' \cap (\mathbb{XA} \cup \mathbb{XB}) \cap \mathrm{N}^{\downarrow})/4 \right\} .$$

An easy calculation gives that there exists an index $i \in [4]$ such that

$$e_{G_\nabla}(C_i, N_i) \geqslant \frac{1}{16} e_{G_\nabla}(\mathbb{H}^*, L'' \cap (\mathbb{XA} \cup \mathbb{XB}) \cap \mathrm{N}^{\downarrow}) \overset{\text{L6.7}}{\geqslant} \frac{1}{128} \tilde{\eta} k n . \tag{6.22}$$

Set $Y := (\mathbb{XA} \cup \mathbb{XB}) \setminus (\mathbb{YB} \cup \mathbb{H}) = (\mathbb{XA} \cup \mathbb{XB}) \setminus \mathbb{YB}$ and $\eta_{\text{L5.2}} = \eta_{\text{L5.3}} := \frac{1}{128}\tilde{\eta}$. By Lemma 3.10 we have

$$|Y| < \frac{\eta_{\text{L5.2}} n}{4\Omega^*} . \tag{6.23}$$

We split the rest of the proof into four subcases according to the value of i.

Subcase B, $i = 1$. We shall apply Lemma 5.2 with the numerical parameters $r_{\text{L5.2}} := 2$, $\Omega^*_{\text{L5.2}} := \Omega^*$, $\Omega^{**}_{\text{L5.2}} := \sqrt{\Omega^{**}}/4$, $\delta_{\text{L5.2}} := \frac{\eta_{\text{L5.2}}\rho^2}{100(\Omega^*)^2}$, $\gamma_{\text{L5.2}} := \rho$, and $\eta_{\text{L5.2}}$, the sets $X_0 := C_1$, $X_1 := N_1$, $X_2 := V(G_{\mathrm{exp}})$, and Y, and the graph $G_{\text{L5.2}}$, which is formed by the vertices of G, with all edges from $E(G_\nabla)$ that are in $E(G_{\mathrm{exp}})$ or that are incident to $\mathbb{H}$. We briefly verify the assumptions of Lemma 5.2. First, the choice of $\delta_{\text{L5.2}}$ guarantees that $(\frac{3\Omega^*_{\text{L5.2}}}{\gamma_{\text{L5.2}}})^2\delta_{\text{L5.2}} < \frac{\eta_{\text{L5.2}}}{10}$. Assumption 1 is given by (6.23). Assumption 2 holds since we assume that (6.22) is satisfied for $i = 1$ and by definition of $\eta_{\text{L5.2}}$. Assumption 3 follows from the definitions of C_1 and of $\mathbb{H}^*$. Assumption 4 follows from the fact that $X_1 \subseteq V(G_{\mathrm{exp}}) = X_2$ and since $\mathrm{mindeg}(G_{\mathrm{exp}}) > \rho k$, which is guaranteed by the definition of a $(k, \Omega^{**}, \Omega^*, \Lambda, \gamma, \varepsilon', \nu, \rho)$-sparse decomposition. This definition also guarantees assumption 5, as $Y \cup X_1 \cup X_2 \subseteq V(G) \setminus \mathbb{H}$.

Lemma 5.2 outputs sets $\mathbb{H}'' := X'_0$, $V_1 := X'_1$, $V_2 := X'_2$ with $\mathrm{mindeg}_{G_\nabla}(\mathbb{H}'', V_1) \geqslant \sqrt[4]{\Omega^{**}}k/2$ (by (d)), $\mathrm{maxdeg}_{G_{\mathrm{exp}}}(V_1, X_2 \setminus V_2) < \rho k/2$ (by (c)), $\mathrm{mindeg}_{G_\nabla}(V_1, \mathbb{H}'') \geqslant \delta_{\text{L5.2}}k$ (by (b)), and $\mathrm{mindeg}_{G_{\mathrm{exp}}}(V_2, V_1) \geqslant \delta_{\text{L5.2}}k$ (by (b)). By (a), we have that $V_1 \subseteq \mathbb{YB} \cap L''$. As $\mathrm{mindeg}_{G_{\mathrm{exp}}}(V_1, X_2) \geqslant \mathrm{mindeg}(G_{\mathrm{exp}}) \geqslant \rho k$, we have $\mathrm{mindeg}_{G_{\mathrm{exp}}}(V_1, V_2) \geqslant \mathrm{mindeg}_{G_{\mathrm{exp}}}(V_1, X_2) - \mathrm{maxdeg}_{G_{\mathrm{exp}}}(V_1, X_2 \setminus V_2) \geqslant \delta_{\text{L5.2}}k$.

Since L', L'', $\mathbb{H}'$ witness preconfiguration $(\clubsuit)(\frac{\tilde{\eta}^3\Omega^{**}}{4\cdot 10^6(\Omega^*)^2})$, this verifies that we have configuration $(\diamond\mathbf{2})(\frac{\tilde{\eta}^3\Omega^{**}}{4\cdot 10^6(\Omega^*)^2}, \sqrt[4]{\Omega^{**}}/2, \frac{\tilde{\eta}\rho^2}{12800(\Omega^*)^2})$.

Subcase B, $i = 2$. We apply Lemma 5.2 with numerical parameters $r_{\text{L5.2}} := 2$, $\Omega^*_{\text{L5.2}} := \Omega^*$, $\Omega^{**}_{\text{L5.2}} := \sqrt{\Omega^{**}}/4$, $\delta_{\text{L5.2}} := \frac{\eta_{\text{L5.2}}\gamma^2}{100(\Omega^*)^2}$, $\gamma_{\text{L5.2}} := \gamma$, and $\eta_{\text{L5.2}}$. Further inputs to the lemma are sets $X_0 := C_2$, $X_1 := N_2$, $X_2 := V(G) \setminus \mathbb{H}$, and Y. The underlying graph $G_{\text{L5.2}}$ is the graph $G_{\mathcal{D}}$ with all edges incident to $\mathbb{H}$ added. Verifying assumptions of Lemma 5.2 is analogous to the verification in Subcase B, $i = 1$, with the exception of assumption 4. To verify this, it suffices to observe that each vertex in X_1 is contained in at least one $(\gamma k, \gamma)$-dense spot from $\mathcal{D}$ (cf. Definition 2.9), and thus has degree at least γk in X_2.

Lemma 5.2 outputs sets X_0', X_1', and X_2' which witness configuration $(\diamond\mathbf{3})(\frac{\tilde{\eta}^3\Omega^{**}}{4\cdot 10^6(\Omega^*)^2}, \sqrt[4]{\Omega^{**}}/2, \gamma/2, \frac{\tilde{\eta}\gamma^2}{12800(\Omega^*)^2})$. In fact, the only thing not analogous to the preceding subcase is that we have to check (4.4). In other words, we have to verify that

$$\mathrm{maxdeg}_{G_{\mathcal{D}}}\big(X_1', V(G)\setminus(X_2'\cup\mathbb{H})\big) \leqslant \frac{\gamma k}{2}\ .$$

As $V(G)\setminus(X_2'\cup\mathbb{H}) = X_2\setminus X_2'$, this follows from (c) of Lemma 5.2.

Subcase B, $i=3$. We apply Lemma 5.2 with numerical parameters $r_{\mathrm{L5.2}} := 3$, $\Omega^*_{\mathrm{L5.2}} := \Omega^*$, $\Omega^{**}_{\mathrm{L5.2}} := \sqrt{\Omega^{**}}/4$, $\delta_{\mathrm{L5.2}} := \frac{\eta_{\mathrm{L5.2}}\gamma^3}{300(\Omega^*)^3}$, $\gamma_{\mathrm{L5.2}} := \gamma$, and $\eta_{\mathrm{L5.2}}$. Further inputs are the sets $X_0 := C_3$, $X_1 := N_3$, $X_2 := \mathbb{E}$, $X_3 := V(G)\setminus\mathbb{H}$, and Y. The underlying graph is $G_{\mathrm{L5.2}} := G_\nabla \cup G_{\mathcal{D}}$. Verifying assumptions of Lemma 5.2 is analogous to the verification in Subcase B, $i=1$, except that for assumption 4 we observe that $\mathrm{mindeg}_{G_\nabla\cup G_{\mathcal{D}}}(X_1,X_2) \geqslant \mathrm{mindeg}_{G_\nabla}(X_1,X_2) \geqslant \gamma k$ by definition of $X_1 = N_3 \subseteq O$, and $\mathrm{mindeg}_{G_\nabla\cup G_{\mathcal{D}}}(X_2,X_3) \geqslant \mathrm{mindeg}_{G_{\mathcal{D}}}(X_2,X_3) \geqslant \gamma k$ for the same reason as in Subcase B, $i=2$.

Lemma 5.2 outputs configuration $(\diamond\mathbf{4})(\frac{\tilde{\eta}^3\Omega^{**}}{4\cdot 10^6(\Omega^*)^2}, \sqrt[4]{\Omega^{**}}/2, \gamma/2, \frac{\tilde{\eta}\gamma^3}{38400(\Omega^*)^3})$, with $\mathbb{H}'' := X_0'$, $V_1 := X_1'$, $\mathbb{E}' := X_2'$, and $V_2 := X_3'$. Indeed, all calculations are similar to those in the preceding two subcases; we need only note additionally that $\mathrm{mindeg}_{G_\nabla\cup G_{\mathcal{D}}}(V_1,\mathbb{E}') \geqslant \frac{\gamma k}{2}\frac{\tilde{\eta}\gamma^3 k}{38400(\Omega^*)^3}$, which follows from the definition of N_3 and of O.

Subcase B, $i=4$. We have that $\mathbf{V}\neq\emptyset$ and $\mathfrak{c}$ is the size of an arbitrary cluster in $\mathbf{V}$. We are going to apply Lemma 5.3 with $\delta_{\mathrm{L5.3}} := \eta_{\mathrm{L5.3}}/100$, $\eta_{\mathrm{L5.3}}$, $h_{\mathrm{L5.3}} := \eta_{\mathrm{L5.3}}\mathfrak{c}/(100\Omega^*)$, $\Omega^*_{\mathrm{L5.3}} := \Omega^*$, $\Omega^{**}_{\mathrm{L5.3}} := \sqrt{\Omega^{**}}/4$ and sets $X_0 := C_4$, $X_1 := N_4$, and Y. The underlying graph is $G_{\mathrm{L5.3}} := G_\nabla$, and $\mathcal{C}_{\mathrm{L5.3}}$ is the set of clusters $\mathbf{V}$.

The fact $e(G)\leqslant kn$ together with (6.22) and the choice of $\eta_{\mathrm{L5.3}}$ gives assumption 2 of Lemma 5.3. The choice of C_4 and $\mathbb{H}^*$ ensures assumption 3. The fact that $X_1\cap\mathbb{H} = \emptyset$ yields assumption 4. With the help of (3.4) it is easy to check assumption 1. Inequality (6.23) implies assumption 5. To verify assumption 6, it is enough to use that $|\mathcal{C}_{\mathrm{L5.3}}| \leqslant \frac{n}{\mathfrak{c}}$. We have thus verified all the assumptions of Lemma 5.3.

We claim that Lemma 5.3 outputs configuration $(\diamond\mathbf{5})(\frac{\tilde{\eta}^3\Omega^{**}}{4\cdot 10^6(\Omega^*)^2}, \sqrt[4]{\Omega^{**}}/2, \frac{\tilde{\eta}}{12800}, \frac{\eta}{2}, \frac{\tilde{\eta}}{12800\Omega^*})$, with $\mathbb{H}'' := X_0'$ and $V_1 := X_1'$. In fact, all conditions of the configuration, except condition (4.12), which we check below, are easy to verify. (Note that $V_1\subseteq \mathbb{YB}$ since $V_1 \subseteq X_1 = N_4 \subseteq \mathbb{XA}\cup\mathbb{XB}$. Also, $V_1\subseteq L''$, and thus V_1 is disjoint from $\mathbb{H}$. Moreover, by the conditions of Lemma 5.3, V_1 is disjoint from Y. So, $V_1\subseteq\mathbb{YB}$.) For (4.12), observe that (6.21) implies that $\mathrm{maxdeg}_{G_\nabla}(N_4,\mathbb{E}) \leqslant \gamma k$. Further, we have $X_1' \subseteq N_4\setminus Y$. So for all $x\in X_1'\subseteq \mathrm{N}^{\downarrow}\setminus Y$, we have that $\deg_{G_\nabla}(x, V(G)\setminus\mathbb{H}) \geqslant \frac{9\eta k}{10}$. As $N_4 \subseteq \bigcup\mathbf{V}\setminus V(G_{\mathrm{exp}})$, we obtain $\deg_{G_{\mathrm{reg}}}(x) \geqslant \frac{9\eta k}{10} - \gamma k \geqslant \frac{\eta k}{2}$, satisfying (4.12).

6.4. Proof of Lemma 6.2. Set $\mathbb{YA}_1' := \{v\in\mathbb{YA}_1 : \deg_{G_{\mathrm{exp}}}(v,\mathbb{YA}_2) \geqslant \rho k\}$. By (6.2) we have

$$e_{G_{\mathrm{exp}}}(\mathbb{YA}_1', \mathbb{YA}_2) \geqslant \rho k n\ . \tag{6.24}$$

Set $r_{\mathrm{L5.4}} := 3$, $\Omega_{\mathrm{L5.4}} := \Omega^*$, $\gamma_{\mathrm{L5.4}} := \frac{\rho\eta}{10^3}$, $\delta_{\mathrm{L5.4}} := \frac{\eta^3\rho^4}{10^{14}(\Omega^*)^3}$, $\eta_{\mathrm{L5.4}} := \rho$. Observe that (5.18) is satisfied for these parameters. Set $Y_{\mathrm{L5.4}} := \bar{V}$, $X_0 := \mathbb{YA}_2$, $X_1 := \mathbb{YA}_1'$,

$X_2 = X_3 := V(G_{\mathrm{exp}})^{\restriction 1}$, and $V := V(G)$. Let $E_2 := E(G_\nabla)$ and $E_1 = E_3 := E(G_{\mathrm{exp}})$. We now briefly verify conditions 1–4 of Lemma 5.4. Condition 1 follows from Definition 3.7(1) and (3.4). Condition 2 follows from (6.24). Using Definition 3.7(6), (3.18), and (3.4), we see that condition 3 for $i = 1$ follows from the definition of $\mathbb{YA}_1'$, and for $i = 2$ from the fact that $\mathrm{mindeg}(G_{\mathrm{exp}}) \geqslant \rho k$. Last, condition 4 follows from the fact that $\bigcup_{i=0}^3 X_i$ is disjoint from $\mathbb{H}$.

Lemma 5.4 yields four nonempty sets $X_0', \dots, X_3'$. By assertions (a), (b), and (c) and hypothesis 3 of Lemma 5.4, for all $i \in \{0, 1, 2, 3\}$, $j \in \{i-1, i+1\} \setminus \{-1, 4\}$, we have

$$\mathrm{mindeg}_{H_{i,j}}(X_i', X_j') \geqslant \delta_{\mathrm{L5.4}} k, \tag{6.25}$$

where $H_{i,j} = G_{\mathrm{exp}}$, except for $\{i, j\} = \{1, 2\}$, where $H_{i,j} = G_\nabla$.

Thus, the sets X_0' and X_1' witness preconfiguration $(\mathbf{exp})(\delta_{\mathrm{L5.4}})$. By Lemma 3.11, and by (6.3) and (6.4), the pair X_0', X_1' together with the cover $\mathcal{F}$ from (3.14) witnesses either preconfiguration $(\heartsuit\mathbf{1})(\frac{3\eta^3}{2\cdot 10^3}, \mathfrak{p}_2(1 + \frac{\eta}{20})k)$ (with respect to $\mathcal{F}$) or preconfiguration $(\heartsuit\mathbf{2})(\mathfrak{p}_2(1 + \frac{\eta}{20})k)$.

Notice that (6.25) establishes properties (4.21)–(4.24). Thus the sets $X_0', \dots, X_3'$ witness configuration $(\diamond\mathbf{6})(\delta_{\mathrm{L5.4}}, 0, 1, 1, \frac{3\eta^3}{2\cdot 10^3}, \mathfrak{p}_2(1 + \frac{\eta}{20})k)$.

6.5. Proof of Lemma 6.3. In Lemmas 6.8, 6.9, 6.11, 6.12, and 6.13 below, we show that cases (**t1**), (**t2**), (**t3**), (**t3–5**), and (**t5**) of Lemma 6.3 lead to configurations $(\diamond\mathbf{6})$, $(\diamond\mathbf{7})$, $(\diamond\mathbf{8})$, $(\diamond\mathbf{9})$, and $(\diamond\mathbf{10})$, respectively. While the first three of these cases are handled by a fairly straightforward application of the cleaning lemma (Lemma 5.5), the latter two cases require some further nontrivial computations.

LEMMA 6.8. *In case* (**t1**) *(of either subcase* (**cA**) *or subcase* (**cB**)*), we obtain configuration* $(\diamond\mathbf{6})(\frac{\eta^3\rho^4}{10^{12}(\Omega^*)^4}, 4\pi, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4}, \frac{3\eta^3}{2000}, \mathfrak{p}_2(1 + \frac{\eta}{20})k)$.

Proof. We use Lemma 5.5 with the following input parameters: $r_{\mathrm{L5.5}} := 3$, $\Omega_{\mathrm{L5.5}} := \Omega^*$, $\gamma_{\mathrm{L5.5}} := \eta\rho/200$, $\eta_{\mathrm{L5.5}} := \rho/(2\Omega^*)$, $\delta_{\mathrm{L5.5}} := \eta^3\rho^4/(10^{12}(\Omega^*)^4)$, $\varepsilon_{\mathrm{L5.5}} := \bar{\varepsilon}$, $\mu_{\mathrm{L5.5}} := \beta$, and $d_{\mathrm{L5.5}} := \bar{d}$. Note that these parameters satisfy the numerical conditions of Lemma 5.5. We use the vertex sets $Y_{\mathrm{L5.5}} := \bar{V} \cup \mathbb{F}$, $X_0 := V_2(\mathcal{M})$, $X_1 := V_1(\mathcal{M})$, $X_2 = X_3 := V(G_{\mathrm{exp}})^{\restriction 1}$, and $V := V(G)$. The partitions of X_0 and X_1 in Lemma 5.5 are the ones induced by $\mathcal{V}(\mathcal{M})$, and the set E_1 consists of all edges from $E(\mathcal{D}_\nabla)$ between pairs from $\mathcal{M}$. Further, we set $E_2 := E(G_\nabla)$ and $E_3 := E(G_{\mathrm{exp}})$.

Let us verify the conditions of Lemma 5.5. Condition 1 follows from Definition 3.7(1) and (3.20). Condition 2 holds by the assumption on $\mathcal{M}$. Condition 3 follows from Definition 3.7(6) by (3.18), and for $i = 1$ also from the definition of $\mathcal{M}$. Condition 4 holds by the definition of $\mathcal{M}$. Finally, condition 5 follows from the properties of the sparse decomposition ∇.

Lemma 5.5 outputs four sets $X_0', \dots, X_3'$. By Lemma 3.11, the sets X_0' and X_1' witness preconfiguration $(\heartsuit\mathbf{1})(\frac{3\eta^3}{2\cdot 10^3}, \mathfrak{p}_2(1 + \frac{\eta}{20})k)$ or $(\heartsuit\mathbf{2})(\mathfrak{p}_2(1 + \frac{\eta}{20})k)$. Further, Lemma 5.5(a) gives that (X_0', X_1') witnesses preconfiguration $(\mathbf{reg})(4\bar{\varepsilon}, \frac{\bar{d}}{4}, \frac{\beta}{2})$. It is now easy to verify that we have configuration $(\diamond\mathbf{6})(\frac{\eta^3\rho^4}{10^{12}(\Omega^*)^4}, 4\bar{\varepsilon}, \frac{\bar{d}}{4}, \frac{\beta}{2}, \frac{3\eta^3}{2\cdot 10^3}, \mathfrak{p}_2(1 + \frac{\eta}{20})k)$.

This leads to configuration $(\diamond\mathbf{6})$ with parameters as claimed. Indeed, no matter whether we have (**M1**) or (**M2**), we have $4\pi \geqslant 4 \cdot \frac{10^5\varepsilon'}{\eta^2}$, $\frac{\gamma^3\rho}{32\Omega^*} \leqslant \frac{\gamma^2}{4}$, and $\frac{\eta^2\nu}{2\cdot 10^4} \leqslant \frac{\eta^2\mathfrak{c}}{8\cdot 10^3 k} \leqslant \frac{\eta^2\varepsilon'}{8\cdot 10^3} \leqslant \frac{\widehat{\alpha}\rho}{\Omega^*}$ (for the latter recall that $\mathfrak{c} \leqslant \varepsilon' k$ by Definition 2.10(4)). $\square$

LEMMA 6.9. *In case* (**t2**) *(of either subcase* (**cA**) *or subcase* (**cB**)*), we obtain configuration* $(\diamond\mathbf{7})(\frac{\eta^3\gamma^3\rho}{10^{12}(\Omega^*)^4}, \frac{\eta\gamma}{400}, 4\pi, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4}, \frac{3\eta^3}{2\cdot 10^3}, \mathfrak{p}_2(1 + \frac{\eta}{20})k)$.

Proof. We use Lemma 5.5 with the following input parameters: $r_{\mathrm{L5.5}} := 3$, $\Omega_{\mathrm{L5.5}} := \Omega^*$, $\gamma_{\mathrm{L5.5}} := \eta\gamma/200$, $\eta_{\mathrm{L5.5}} := \rho/\Omega^*$, $\delta_{\mathrm{L5.5}} := \eta^3\gamma^3\rho/(10^{12}(\Omega^*)^4)$, $\varepsilon_{\mathrm{L5.5}} := \bar{\varepsilon}$, $\mu_{\mathrm{L5.5}} := \beta$, and $d_{\mathrm{L5.5}} := \bar{d}$. We use the vertex sets $Y_{\mathrm{L5.5}} := \bar{V} \cup \mathbb{F}$, $X_0 := V_2(\mathcal{M})$, $X_1 := V_1(\mathcal{M})$, $X_2 := \mathbb{E}^{\restriction 1}$, $X_3 := \mathbb{A}_1$, and $V := V(G)$. The partitions of X_0 and X_1 in Lemma 5.5 are the ones induced by $\mathcal{V}(\mathcal{M})$, and the set E_1 consists of all edges from $E(\mathcal{D}_\nabla)$ between pairs from $\mathcal{M}$. Further, we set $E_2 := E(G_\nabla)$ and $E_3 := E(G_\mathcal{D})$.

The conditions of Lemma 5.5 are verified as before; let us just note that condition 3 follows from Definition 3.7(6) and by (3.18), and for $i = 1$ from the definition of $\mathcal{M}$, while for $i = 2$ it holds since $\mathbb{E}$ is covered by the set $\mathcal{D}$ of $(\gamma k, \gamma)$-dense spots (cf. Definition 2.9).

It is now easy to check that the output of Lemma 5.5 are sets that witness configuration $(\diamond\mathbf{7})(\frac{\eta^3\gamma^3\rho}{10^{12}(\Omega^*)^4}, \frac{\eta\gamma}{400}, 4\bar{\varepsilon}, \frac{\bar{d}}{4}, \frac{\beta}{2}, \frac{3\eta^3}{2\cdot 10^3}, \mathfrak{p}_2(1+\frac{\eta}{20})k)$. ∎

Before proceeding with dealing with cases (**t3**), (**t5**), and (**t3**–**5**) we state some properties of the matching $\bar{\mathcal{M}} := (\mathcal{M}_A \cup \mathcal{M}_B)^{\restriction 1}$.

Lemma 6.10. *For* $V_{\mathrm{leftover}} := V(\mathcal{M}_A \cup \mathcal{M}_B)^{\restriction 1} \setminus V(\bar{\mathcal{M}})$ *and* $Y_{\bar{\mathcal{M}}} := \bar{V} \cup \mathbb{F} \cup \mathrm{shadow}_{G_\mathcal{D}}(V_{\mathrm{leftover}}, \frac{\eta^2 k}{1000})$*, we have*

(a) $\bar{\mathcal{M}}$ *is a* $(\frac{400\varepsilon}{\eta}, \frac{d}{2}, \frac{\eta\pi\mathfrak{c}}{200})$*-regularized matching absorbed by* $\mathcal{M}_A \cup \mathcal{M}_B$ *and* $V(\bar{\mathcal{M}}) \subseteq \mathbb{A}_1$*, and*

(b) $|Y_{\bar{\mathcal{M}}}| \leqslant \frac{3000\varepsilon\Omega^* n}{\eta^2}$.

Proof. Lemma 6.10(a) follows from Lemma 3.9.

Observe that from properties (1) and (3) of Definition 3.7 we can calculate that

(6.26)
$$|V_{\mathrm{leftover}}| \leqslant 3\cdot k^{0.9}\cdot|\mathcal{M}_A\cup\mathcal{M}_B| + \left|\bigcup\bar{\mathcal{V}}\cup\bar{\mathcal{V}}^*\right| \leqslant 3\cdot k^{0.9}\cdot\frac{n}{2\pi\mathfrak{c}} + 2\exp(-k^{0.1}) \overset{(3.4)}{\leqslant} 2\varepsilon n\ .$$

Then

$$\begin{aligned} |Y_{\bar{\mathcal{M}}}| &\leqslant |\bar{V}| + |\mathbb{F}| + \left|\mathrm{shadow}_{G_\mathcal{D}}\left(V_{\mathrm{leftover}}, \frac{\eta^2 k}{1000}\right)\right| \\ \text{(by F3.1)} \quad &\leqslant |\bar{V}| + |\mathbb{F}| + |V_{\mathrm{leftover}}|\frac{1000\Omega^*}{\eta^2} \\ \text{(by (6.26), D3.7(1), (3.4), (3.20))} \quad &< \frac{3000\varepsilon\Omega^* n}{\eta^2}\ , \end{aligned}$$

as desired for Lemma 6.10(b). ∎

Lemma 6.11. *In Case* (**t3**)(**cA**) *we obtain configuration* $(\diamond\mathbf{8})(\frac{\eta^4\gamma^4\rho}{10^{15}(\Omega^*)^5}, \frac{\eta\gamma}{400}, \frac{400\varepsilon}{\eta}, 4\bar{\varepsilon}, \frac{d}{2}, \frac{\bar{d}}{4}, \frac{\eta\pi\mathfrak{c}}{200k}, \frac{\beta}{2}, \mathfrak{p}_1(1+\frac{\eta}{20})k, \mathfrak{p}_2(1+\frac{\eta}{20})k)$.

Proof. We use Lemma 5.5 with the following input parameters: $r_{\mathrm{L5.5}} := 4$, $\Omega_{\mathrm{L5.5}} := \Omega^*$, $\gamma_{\mathrm{L5.5}} := \eta\gamma/200$, $\eta_{\mathrm{L5.5}} := \rho/\Omega^*$, $\delta_{\mathrm{L5.5}} := \eta^4\gamma^4\rho/(10^{15}(\Omega^*)^5)$, $\varepsilon_{\mathrm{L5.5}} := \bar{\varepsilon}$, $\mu_{\mathrm{L5.5}} := \beta$, and $d_{\mathrm{L5.5}} := \bar{d}$. We use the following vertex sets: $Y_{\mathrm{L5.5}} := Y_{\bar{\mathcal{M}}}$, $X_0 := V_2(\mathcal{M})$, $X_1 := V_1(\mathcal{M})$,

$$X_2 := (\mathbb{L}_{\eta,k}(G)\cap V_{\rightsquigarrow\mathbb{E}})^{\restriction 0} \setminus \left(V(G_{\mathrm{exp}})\cup\mathbb{E}\cup V(\mathcal{M}_A\cup\mathcal{M}_B)\cup V_{\rightsquigarrow\mathbb{H}}\cup L_\#\cup\mathbb{J}_\mathbb{E}\cup\mathbb{J}_1\right)\ ,$$

$X_3 := \mathbb{E}^{\restriction 1}$, $X_4 := \mathbb{A}_1$, and $V := V(G)$. The partitions $P_i^{(j)}$ of X_0 and X_1 in Lemma 5.5 are the ones induced by $\mathcal{V}(\mathcal{M})$, and the set E_1 consists of all edges from $E(\mathcal{D}_\nabla)$ between pairs from $\mathcal{M}$. Further, we set $E_2 = E_3 := E(G_\nabla)$ and $E_4 := E(G_\mathcal{D})$.

Most of the conditions of Lemma 5.5 are verified as before; let us only note the few differences. Condition 1 follows from Lemma 6.10(b). Using Definition 3.7(6) and (3.18), we find that condition 3 for $i = 2$ follows from the definition of $V_{\rightsquigarrow\mathbb{E}}$, and condition 3 for $i = 3$ holds, as it is the same as condition 3 for $i = 2$ in Lemma 6.9. To verify condition 3 for $i = 1$, we first observe that since we are in case (**t3**), we have

$$V_1(\mathcal{M}) \subseteq \mathrm{shadow}_{G_\nabla}\left((V_{\rightsquigarrow\mathbb{E}} \cap \mathbb{L}_{\eta,k}(G)) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B), \frac{2\eta^2 k}{10^5}\right) \setminus \left(\mathrm{shadow}_{G_\nabla}(V(G_{\mathrm{exp}}), \rho k) \cup V_{\rightsquigarrow\mathbb{E}}\right) . \tag{6.27}$$

Also, since we are in case (**cA**), we have

$$V_1(\mathcal{M}) \cap \mathbb{J} = \emptyset . \tag{6.28}$$

Thus, for each $v \in V_1(\mathcal{M})$ we have, using Definition 3.7(6),

$$\begin{aligned}
\deg_{G_\nabla}(v, X_2) &\geqslant \mathfrak{p}_0\Big(\deg_{G_\nabla}(v, (\mathbb{L}_{\eta,k}(G) \cap V_{\rightsquigarrow\mathbb{E}}) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)) \\
&\qquad - \deg_{G_\nabla}(v, V(G_{\mathrm{exp}}) \cup \mathbb{E} \cup V_{\rightsquigarrow\mathbb{H}} \cup L_\# \cup \mathbb{J}_\mathbb{E} \cup \mathbb{J}_1)\Big) - k^{0.9} \\
\text{(by (6.27), (6.28), (3.18))} \quad &\geqslant \frac{\eta}{100}\left(\frac{2\eta^2 k}{10^5} - \rho k - \frac{\rho k}{100\Omega^*} - \frac{\eta^2 k}{10^5}\right) - k^{0.9} \\
\text{(by (3.4))} \quad &\geqslant \frac{\eta\gamma k}{200} ,
\end{aligned}$$

which indeed verifies condition 3 for $i = 1$.

Define $\mathcal{N} := \bar{\mathcal{M}} \setminus \{(X, Y) \in \bar{\mathcal{M}} \,:\, X \cup Y \subseteq V(\mathcal{N}_\mathbb{E})\}$. By Lemma 6.10(a) we have that $\mathcal{N} \subseteq \bar{\mathcal{M}}$ is a $(\frac{400\varepsilon}{\eta}, \frac{d}{2}, \frac{\eta\pi\mathfrak{c}}{200})$-regularized matching absorbed by $\mathcal{M}_A \cup \mathcal{M}_B$ and that $V(\mathcal{N}) \subseteq \mathbb{A}_1$.

To see that the output of Lemma 5.5 together with the matching $\mathcal{N}$ leads to configuration $(\diamond\mathbf{8})(\frac{\eta^4\gamma^4\rho}{10^{15}(\Omega^*)^5}, \frac{\eta\gamma}{400}, \frac{400\varepsilon}{\eta}, 4\bar{\varepsilon}, \frac{d}{2}, \frac{\bar{d}}{4}, \frac{\eta\pi\mathfrak{c}}{200k}, \frac{\beta}{2}, \mathfrak{p}_1(1+\frac{\eta}{20})k, \mathfrak{p}_2(1+\frac{\eta}{20})k)$, let us show that (4.35) is satisfied (the other conditions are more easily seen to hold).

For this, let $v \in X_2'$. We have to show that

$$\deg_{G_\mathcal{D}}(v, X_3') + \deg_{G_{\mathrm{reg}}}(v, V(\mathcal{N})) \geqslant \mathfrak{p}_1\left(1 + \frac{\eta}{20}\right)k. \tag{6.29}$$

Note that $v \notin V(G_{\mathrm{exp}})$, and thus $\deg_{G_{\mathrm{exp}}}(v) = 0$. This allows us to calculate as follows:

$$\begin{aligned}
\deg_{G_\mathcal{D}}(v, X_3') + \deg_{G_{\mathrm{reg}}}(v, V(\mathcal{N})) &\geqslant \deg_{G_\nabla}(v, \mathbb{A}_1) - \deg_{G_\mathcal{D}}(v, X_3 \setminus X_3') \\
&\quad - \deg_{G_{\mathrm{reg}}}(v, V(\mathcal{N}_\mathbb{E})) - \deg_{G_{\mathrm{reg}}}(v, V_{\mathrm{leftover}}) \\
&\quad - \deg_{G_{\mathrm{reg}}}(v, V(G) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)) .
\end{aligned} \tag{6.30}$$

We now bound the terms of the right-hand side of (6.30). From Definition 3.7(6) we obtain that $\deg_{G_\nabla}(v, \mathbb{A}_1) \geqslant \mathfrak{p}_1(\deg_{G_\nabla}(v) - \deg_G(v, \mathbb{H})) - k^{0.9}$. Lemma 5.5(c) gives that $\deg_{G_\mathcal{D}}(v, X_3 \setminus X_3') \leqslant \frac{\eta\gamma k}{400}$. As $v \notin \mathbb{J}_\mathbb{E} \cup V(\mathcal{M}_A \cup \mathcal{M}_B)$, we have $\deg_{G_{\mathrm{reg}}}(v, V(\mathcal{N}_\mathbb{E})) < \gamma k$. As $v \notin Y_{\bar{\mathcal{M}}}$ and thus $v \notin \mathrm{shadow}_{G_\mathcal{D}}(V_{\mathrm{leftover}}, \frac{\eta^2 k}{1000})$, we have $\deg_{G_\mathcal{D}}(v, V_{\mathrm{leftover}}) \leqslant \frac{\eta^2 k}{1000}$. Last, recall that $v \notin \mathbb{J}_1 \cup V(\mathcal{M}_A \cup \mathcal{M}_B)$, and consequently $\deg_{G_{\mathrm{reg}}}(v, V(G) \setminus$

$V(\mathcal{M}_A \cup \mathcal{M}_B)) < \gamma k$. Putting these bounds together, we find that

$$\begin{aligned}
\deg_{G_{\mathcal{D}}}(v, X_3') + \deg_{G_{\mathrm{reg}}}(v, V(\mathcal{N})) &\geqslant \mathfrak{p}_1 \left(\deg_{G_\nabla}(v) - \deg_G(v, \mathbb{H})\right) - \frac{2\eta^2 k}{1000} \\
\text{(as } v \in \mathbb{L}_{\eta,k}(G) \setminus (L_{\#} \cup V_{\rightsquigarrow\mathbb{H}})\text{)} \quad &\geqslant \mathfrak{p}_1 \left(\left(1 + \frac{9\eta}{10}\right) k - \frac{\eta k}{100}\right) - \frac{\eta^2 k}{500} \\
\text{(by (3.18), (3.4))} \quad &\geqslant \mathfrak{p}_1 \left(1 + \frac{\eta}{20}\right) k \, .
\end{aligned}$$

This proves (6.29). ∎

LEMMA 6.12. *In case* (**t3**–**5**)(**cB**) *we get configuration* $(\diamond\mathbf{9})(\frac{\rho\eta^8}{10^{27}(\Omega^*)^3}, \frac{2\eta^3}{10^3}, \mathfrak{p}_1(1 + \frac{\eta}{40})k, \mathfrak{p}_2(1 + \frac{\eta}{20})k, \frac{400\varepsilon}{\eta}, \frac{d}{2}, \frac{\eta\pi\mathfrak{c}}{200k}, 4\pi, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4})$.

Proof. Recall that by Lemma 3.11 we know that $\mathcal{F}$, as defined in (3.14), is an $(\mathcal{M}_A \cup \mathcal{M}_B)$-cover. We introduce another $(\mathcal{M}_A \cup \mathcal{M}_B)$-cover,

$$\mathcal{F}' := \mathcal{F} \cup \{X \in \mathcal{V}(\mathcal{M}_B) \; : \; X \subseteq \mathbb{E}\} \, .$$

By (3.32) and as we are in case (**cB**), we have $\mathrm{maxdeg}_{G_\nabla}(V_1(\mathcal{M}), \bigcup \mathcal{F}) \leqslant \frac{2\eta^3}{3\cdot 10^3}k$. Furthermore, as we are in case (**t3**–**5**), we have $V_1(\mathcal{M}) \cap V_{\rightsquigarrow\mathbb{E}} = \emptyset$. Thus,

$$\mathrm{maxdeg}_{G_\nabla}\left(V_1(\mathcal{M}), \bigcup \mathcal{F}'\right) \leqslant \frac{2\eta^3}{10^3} k \, . \tag{6.31}$$

We use Lemma 5.5 with the following input parameters: $r_{\mathrm{L5.5}} := 2$, $\Omega_{\mathrm{L5.5}} := \Omega^*$, $\gamma_{\mathrm{L5.5}} := \eta^4/10^{11}$, $\eta_{\mathrm{L5.5}} := \rho/2\Omega^*$, $\delta_{\mathrm{L5.5}} := \rho\eta^8/(10^{27}(\Omega^*)^3)$, $\varepsilon_{\mathrm{L5.5}} := \bar{\varepsilon}$, $\mu_{\mathrm{L5.5}} := \beta$, and $d_{\mathrm{L5.5}} := \bar{d}$. We use the following vertex sets: $Y_{\mathrm{L5.5}} := Y_{\bar{\mathcal{M}}}$, $X_0 := V_2(\mathcal{M})$, $X_1 := V_1(\mathcal{M})$, and $X_2 := V(\bar{\mathcal{M}}) \setminus \bigcup \mathcal{F}' \subseteq \bigcup \mathbf{V}^{\upharpoonright 1}$. The partitions of X_0 and X_1 in Lemma 5.5 are the ones induced by $\mathcal{V}(\mathcal{M})$, and the set E_1 consists of all edges from $E(\mathcal{D}_\nabla)$ between pairs from $\mathcal{M}$. Further, we set $E_2 := E(G_{\mathcal{D}})$.

Condition 1 of Lemma 5.5 follows from Lemma 6.10(b). Condition 2 follows by the assumption of Lemma 6.12 on the size of $V(\mathcal{M})$. Condition 4 follows from the definition of $\mathcal{M}$. Condition 5 holds since $V(\mathcal{M})$ does not meet $\mathbb{H}$.

It remains to verify condition 3 for $i = 1$. For this, first note that from Lemma 3.11 we get that

$$\mathrm{mindeg}_{G_\nabla}\left(V_1(\mathcal{M}), V_{\mathrm{good}}^{\upharpoonright 1}\right) \overset{(\mathbf{cB})}{\geqslant} \mathrm{mindeg}_{G_\nabla}\left(\mathbb{XA} \setminus (\mathbb{J} \cup \bar{V}), V_{\mathrm{good}}^{\upharpoonright 1}\right) \geqslant \mathfrak{p}_1 \left(1 + \frac{\eta}{20}\right) k \, . \tag{6.32}$$

From this, we calculate that

$$\begin{aligned}
&\mathrm{mindeg}_{G_{\mathcal{D}}}\big(V_1(\mathcal{M}), V(\mathcal{M}_A \cup \mathcal{M}_B)^{\upharpoonright 1}\big) \\
&\quad \geqslant \mathrm{mindeg}_{G_\nabla}\big(V_1(\mathcal{M}), V(\mathcal{M}_A \cup \mathcal{M}_B)^{\upharpoonright 1}\big) \\
&\qquad - \mathrm{maxdeg}_{G_{\mathrm{exp}}}\big(V_1(\mathcal{M}), V(\mathcal{M}_A \cup \mathcal{M}_B)\big) \\
\text{(by (3.10), (3.7))} &\quad \geqslant \mathrm{mindeg}_{G_\nabla}\big(V_1(\mathcal{M}), V_{\mathrm{good}}^{\upharpoonright 1}\big) \\
&\qquad - \mathrm{maxdeg}_{G_\nabla}\big(V_1(\mathcal{M}), \mathbb{E}\big) \\
&\qquad - \mathrm{maxdeg}_{G_\nabla}\big(V_1(\mathcal{M}), \mathbb{L}_{\eta,k}(G) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)\big) \\
&\qquad - \mathrm{maxdeg}_{G_\nabla}\big(V_1(\mathcal{M}), V(G_{\mathrm{exp}}) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)\big) \\
&\qquad - \mathrm{maxdeg}_{G_\nabla}\big(V_1(\mathcal{M}), V(G_{\mathrm{exp}}) \cap V(\mathcal{M}_A \cup \mathcal{M}_B)\big) \, .
\end{aligned}$$

We use (6.32) to get a lower bound on the first term. Recalling that $V_1(\mathcal{M})\cap V_{\rightsquigarrow\mathbb{E}}=\emptyset$, we obtain an upper bound of $\frac{\rho k}{100\Omega^*}$ on the second term. Last, using (**cB**) we can bound $\mathrm{maxdeg}_{G_\nabla}(V_1(\mathcal{M}),\mathbb{L}_{\eta,k}(G)\setminus V(\mathcal{M}_A\cup\mathcal{M}_B))$ by $\mathrm{maxdeg}_{G_\nabla}(\mathbb{XA}\setminus\mathbb{J}_3,\mathbb{XA})$. Therefore,

$$\begin{aligned}\mathrm{mindeg}_{G_\mathcal{D}}\big(V_1(\mathcal{M}),V(\mathcal{M}_A\cup\mathcal{M}_B)^{\restriction 1}\big)\geqslant \mathfrak{p}_1\left(1+\frac{\eta}{20}k\right)-\frac{\rho k}{100\Omega^*}\\ -\mathrm{maxdeg}_{G_\nabla}\big(\mathbb{XA}\setminus\mathbb{J}_3,\mathbb{XA}\big)\\ -\mathrm{maxdeg}_{G_\nabla}(V_1(\mathcal{M}),V(G_{\mathrm{exp}}))\ .\end{aligned}$$

We use the definition of $\mathbb{J}_3$ and the fact that (**t3–5**) gives

$$V_1(\mathcal{M})\cap\mathrm{shadow}_G(V(G_{\mathrm{exp}}),\rho k)=\emptyset.$$

We obtain

$$\mathrm{mindeg}_{G_\mathcal{D}}\big(V_1(\mathcal{M}),V(\mathcal{M}_A\cup\mathcal{M}_B)^{\restriction 1}\big)\geqslant \mathfrak{p}_1\left(1+\frac{\eta}{20}\right)k-\frac{\rho k}{100\Omega^*}-\frac{\eta^3k}{10^3}-\rho k\ . \tag{6.33}$$

Therefore,

$$\begin{aligned}\mathrm{mindeg}_{G_\mathcal{D}}(V_1(\mathcal{M})\setminus Y_{\mathrm{L5.5}},X_2)&\geqslant \mathrm{mindeg}_{G_\mathcal{D}}\big(V_1(\mathcal{M})\setminus Y_{\bar M},V(\bar{\mathcal{M}})\big)\\ &\quad-\mathrm{maxdeg}_{G_\mathcal{D}}\Big(V_1(\mathcal{M}),\bigcup\mathcal{F}'\Big)\\ \text{(by def of }\bar{\mathcal{M}}\text{, (6.31))}\quad&\geqslant \mathrm{mindeg}_{G_\mathcal{D}}\big(V_1(\mathcal{M}),V(\mathcal{M}_A\cup\mathcal{M}_B)^{\restriction 1}\big)\\ &\quad-\mathrm{maxdeg}_{G_\mathcal{D}}(V_1(\mathcal{M})\setminus Y_{\bar M},V_{\mathrm{leftover}})-\frac{2\eta^3k}{10^3}\\ \text{(by (6.33) and by def of }Y_{\mathrm{L5.5}})\quad&\geqslant \mathfrak{p}_1\left(1+\frac{\eta}{20}\right)k-\frac{\rho k}{100\Omega^*}-\frac{\eta^3k}{10^3}-\rho k-\frac{\eta^2k}{1000}-\frac{2\eta^3k}{10^3}\\ &\geqslant \mathfrak{p}_1(1+\frac{\eta}{30})k\ . \end{aligned}\tag{6.34}$$

Since the last term is greater than $\gamma_{\mathrm{L5.5}}k=\frac{\eta^4}{10^{11}}k$ by (3.18), we see that condition 3 of Lemma 5.5 is satisfied.

Lemma 5.5 outputs three nonempty sets X_0',X_1',X_2' disjoint from $Y_{\mathrm{L5.5}}$, together with $(4\bar\varepsilon,\frac{\bar d}{4})$-superregular pairs $\{Q_0^{(j)},Q_1^{(j)}\}_{j\in\mathcal{Y}}$ which cover (X_0',X_1') with the following properties:

$$\text{(by L5.5(a))}\quad \min\Big\{|Q_0^{(j)}|,|Q_1^{(j)}|\Big\}\geqslant\frac{\beta k}{2}\quad\text{for each } j\in\mathcal{Y}\ , \tag{6.35}$$

$$\text{(by L5.5(b))}\quad \mathrm{mindeg}_{G_\mathcal{D}}(X_2',X_1')\geqslant\delta_{\mathrm{L5.5}}k\ , \tag{6.36}$$

$$\begin{aligned}\text{(by L5.5(c) and (6.34))}\quad \mathrm{mindeg}_{G_\mathcal{D}}(X_1',X_2')&\geqslant\mathfrak{p}_1\Big(1+\frac{\eta}{30}\Big)k-\frac{\eta^4k}{2\cdot10^{11}}\\ &\geqslant\mathfrak{p}_1\Big(1+\frac{\eta}{40}\Big)k\ .\end{aligned}\tag{6.37}$$

We now verify that the sets X_0',X_1',X_2', the regularized matching $\mathcal{N}_{\mathrm{D4.14}}:=\bar{\mathcal{M}}$ together with the $(\mathcal{M}_A\cup\mathcal{M}_B)$-cover $\mathcal{F}'$, and the family $\{(Q_0^{(j)},Q_0^{(j)})\}_{j\in\mathcal{Y}}$ satisfy all the conditions of configuration $(\diamond\mathbf{9})(\delta_{\mathrm{L5.5}},\frac{2\eta^3}{10^3},\mathfrak{p}_1(1+\frac{\eta}{40})k,\mathfrak{p}_2(1+\frac{\eta}{20})k,\frac{400\varepsilon}{\eta},\frac{d}{2},\frac{\eta\pi\mathfrak{c}}{200k},4\pi,\frac{\gamma^3\rho}{32\Omega^*},\frac{\eta^2\nu}{2\cdot10^4})$.

By Lemma 3.11, since we are in case (**cB**), and by (6.31), the pair X_0',X_1' together with the $(\mathcal{M}_A\cup\mathcal{M}_B)$-cover $\mathcal{F}'$ witnesses preconfiguration $(\heartsuit\mathbf{1})(\frac{2\eta^3}{10^3},\mathfrak{p}_2(1+\frac{\eta}{20})k)$. By Lemma 6.10(a), $\bar{\mathcal{M}}$ is as required for configuration $(\diamond\mathbf{9})$.

To see that G is in preconfiguration $(\mathbf{reg})(4\pi, \gamma^3\rho/32\Omega^*, \eta^2\nu/2\cdot 10^4)$, note that $4\bar{\varepsilon} \leqslant 4\pi$ and $\bar{d}/4 \geqslant \gamma^3\rho/32\Omega^*$ (in both cases (**M1**) and (**M2**)). Further, property (4.20) follows from (6.35) since $\beta/2 \geqslant \eta^2\nu/2\cdot 10^4$.

Finally, by definition of X_2, the set X_2' is as required, with property (4.36) following from (6.37), and property (4.37) following from (6.36). □

We are now reaching the last lemma of this section, dealing with the last remaining case.

LEMMA 6.13. *Finally, in case* (**t5**)(**cA**) *we get configuration* $(\diamond\mathbf{10})(\varepsilon, \frac{\gamma^2 d}{2}, \pi\sqrt{\varepsilon'}\nu k, \frac{(\Omega^*)^2 k}{\gamma^2}, \frac{\eta}{40})$.

Proof. Since we are in case (**t5**), we have $V(\mathcal{M}) \subseteq V(G_{\mathrm{reg}})$. Therefore,

$$\begin{aligned}\mathrm{mindeg}_{G_{\mathrm{reg}}}(V(\mathcal{M}), V_{\mathrm{good}}) &\geqslant \mathrm{mindeg}_{G_\nabla}(V(\mathcal{M}), V_+ \setminus L_\#) - \mathrm{maxdeg}_{G_\nabla}(V(\mathcal{M}), \mathbb{H}) \\ &\quad - \mathrm{maxdeg}_{G_\nabla}(V(\mathcal{M}), \mathbb{E}) - \mathrm{maxdeg}_{G_\nabla}(V(\mathcal{M}), V(G_{\exp})) \\ &\geqslant (1+\frac{\eta}{20})k, \end{aligned} \tag{6.38}$$

where the last line follows as $V(\mathcal{M}) \subseteq \mathbb{XA}\setminus \mathbb{J} \subseteq \mathbb{YA}\setminus V_{\rightsquigarrow\mathbb{H}}$ by (**cA**), and furthermore, $V(\mathcal{M}) \cap (\mathrm{shadow}_G(V(G_{\exp}), \rho k) \cup V_{\rightsquigarrow\mathbb{E}}) = \emptyset$ by (**t5**).

Define

$$\begin{aligned}\mathcal{C} &:= \left\{C \setminus \left(L_\# \cup V(\mathcal{M}_A \cup \mathcal{M}_B) \cup V_{\rightsquigarrow\mathbb{H}} \cup \mathbb{J}_1\right) \;:\; C \in \mathbf{V}\right\} ,\\ \mathcal{C}^- &:= \left\{C \in \mathcal{C} \;:\; |C| < \sqrt{\varepsilon'}\mathfrak{c}\right\} .\end{aligned}$$

We have

$$\left|\bigcup \mathcal{C}^-\right| \leqslant \sum_{C\in\mathcal{C}} \sqrt{\varepsilon'}|C| \leqslant \sqrt{\varepsilon'} n . \tag{6.39}$$

Set $\mathcal{V}^\circ := \mathcal{V}(\mathcal{M}_A \cup \mathcal{M}_B) \cup (\mathcal{C}\setminus\mathcal{C}^-)$, and let G° be the subgraph of G with vertex set $\bigcup\mathcal{V}^\circ$ and all edges from $E(G_{\mathrm{reg}})$ induced by $\bigcup\mathcal{V}^\circ$ plus all edges of $E(G_\nabla)\setminus E(G_{\exp})$ between X and Y for all $(X,Y) \in \mathcal{M}_A\cup\mathcal{M}_B$. Apply Fact 2.1 (and recall Definition 2.10(3)) to see that each pair of sets $X, Y \in \mathcal{V}^\circ$ forms an ε-regular pair of density either 0 or at least $\gamma^2 d/2$ (whose edges either lie in G_{reg} or touch $\mathbb{E}$).

Next, observe that from Setting 3.5(3), Facts 2.7 and 2.8, and using Definition 2.10(7), we find that for all $X \in \mathcal{V}^\circ$ which lie in some cluster of $\mathbf{V}$, we have $|\bigcup \mathrm{N}_{G^\circ}(X)| \leqslant |\bigcup \mathrm{N}_{G_{\mathcal{D}}}(X)| \leqslant \frac{\Omega^*}{\gamma}\cdot\frac{\Omega^* k}{\gamma}$. Also, observe that for all $X\in\mathcal{V}^\circ$ which do not lie in some cluster of $\mathbf{V}$, we know from Setting 3.5(4) that X is not incident to any edges from $E(G_{\mathrm{reg}})$. This means that $\bigcup\mathrm{N}_{G^\circ}(X)$ is contained in the partner of X in $\mathcal{M}_A \cup M_B$ (which has size at most $\mathfrak{c} \leqslant \varepsilon' k$ by Setting 3.5(4) and Definition 2.10(4)).

Thus we obtain that

$$(G^\circ, \mathcal{V}^\circ) \text{ is an } \left(\varepsilon, \frac{\gamma^2 d}{2}, \pi\sqrt{\varepsilon'}\mathfrak{c}, \frac{(\Omega^*)^2 k}{\gamma^2}\right)\text{-regularized graph} . \tag{6.40}$$

Define

$$\mathcal{L}^\circ := \left\{X \in \mathcal{V}^\circ\setminus\mathcal{V}(\mathcal{M}_A\cup\mathcal{M}_B) \;:\; \mathrm{mindeg}_{G^\circ}(X) \geqslant \left(1+\frac{\eta}{2}\right)k\right\} .$$

We claim that the following holds.

Claim 6.13.1. *There are distinct* $X_A, X_B \in \mathcal{V}^\circ$, *with* $E(G^\circ[X_A, X_B]) \neq \emptyset$, *such that we have* $\deg_{G_{\mathrm{reg}}}(v, V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \bigcup \mathcal{L}^\circ) \geqslant (1 + \frac{\eta}{40})k$ *for all but at most* $2\varepsilon'\mathfrak{c}$ *vertices* $v \in X_A$, *and all but at most* $2\varepsilon'\mathfrak{c}$ *vertices* $v \in X_B$.

Then, setting $\tilde{G}_{\mathrm{D4.16}} := G^\circ$, $\mathcal{V}_{\mathrm{D4.16}} := \mathcal{V}^\circ$, $\mathcal{M}_{\mathrm{D4.16}} := \mathcal{M}_A \cup \mathcal{M}_B$, $\mathcal{L}^*_{\mathrm{D4.16}} := \mathcal{L}^\circ$, $A_{\mathrm{D4.16}} := X_A$, and $B_{\mathrm{D4.16}} := X_B$, we obtain configuration $(\diamond\mathbf{10})(\varepsilon, \frac{\gamma^2 d}{2}, \pi\sqrt{\varepsilon'}\nu k, \frac{(\Omega^*)^2 k}{\gamma^2}, \frac{\eta}{40})$. Indeed, using (6.40) and the definition of $\mathcal{L}^\circ$, we see that $(\tilde{G}_{\mathrm{D4.16}}, \mathcal{V}_{\mathrm{D4.16}})$, $\mathcal{M}_{\mathrm{D4.16}}$, and $\mathcal{L}^*_{\mathrm{D4.16}}$ are as desired and fulfill (c) of Definition 4.16. Claim 6.13.1 together with the fact that $\deg_{G^\circ}(v, V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \bigcup \mathcal{L}^\circ) \geqslant \deg_{G_{\mathrm{reg}}}(v, V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \bigcup \mathcal{L}^\circ)$ for all $v \in V(G^\circ)$ ensures that also (a) and (b) of Definition 4.16 hold.

It remains only to prove Claim 6.13.1.

Proof of Claim 6.13.1. In order to find X_A and X_B as in the statement of the claim, we shall exploit the matching $\mathcal{M}$; the relation between $\mathcal{M}$ and $(G^\circ, \mathcal{V}^\circ)$, $\mathcal{M}_A \cup \mathcal{M}_B$, and $\mathcal{L}^\circ$ is not direct. We proceed as follows. In Subclaim 6.13.1.1 we find a suitable $\mathcal{M}$-edge. In case (**M1**) this $\mathcal{M}$-edge readily gives a suitable pair $(A_{\mathrm{D4.16}}, B_{\mathrm{D4.16}})$. In case (**M2**) we have to work on the $\mathcal{M}$-edge to get a suitable $\mathbf{G}_{\mathrm{reg}}$-edge; this will be done in Subclaim 6.13.1.2. Only then do we find $(A_{\mathrm{D4.16}}, B_{\mathrm{D4.16}})$.

Subclaim 6.13.1.1. *There is an* $\mathcal{M}$-*edge* (A, B) *such that* $\deg_{G_{\mathrm{reg}}}(v, V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \bigcup \mathcal{L}^\circ) \geqslant (1 + \frac{\eta}{40})k + \frac{\eta k}{200}$ *for at least* $\frac{|A|}{2}$ *vertices* $v \in A$, *and at least* $\frac{|B|}{2}$ *vertices* $v \in B$.

Proof of Subclaim 6.13.1.1. Set $S := \mathrm{shadow}_{G_{\mathrm{reg}}}(\bigcup \mathcal{C}^-, \frac{\eta k}{200})$, and note that by Fact 3.1 we have $|S| \leqslant |\bigcup \mathcal{C}^-| \cdot \frac{200\Omega^*}{\eta}$. So, setting $\mathcal{M}_S := \{(X,Y) \in \mathcal{M} : |(X \cup Y) \cap S| \geqslant \frac{|X \cup Y|}{4}\}$, we find that

$$|V(\mathcal{M}_S)| \;\leqslant\; 4|S| \overset{(6.39)}{\leqslant} \frac{800\sqrt{\varepsilon'}\Omega^* n}{\eta} \;<\; \frac{\rho n}{\Omega^*} \;\leqslant\; |V(\mathcal{M})| \,,$$

where the last inequality holds by the assumption of Lemma 6.13. Consequently, $\mathcal{M} \neq \mathcal{M}_S$.

Let $(A, B) \in \mathcal{M} \setminus \mathcal{M}_S$. We will show that (A, B) satisfies the requirements of the subclaim. We start by proving that

$$V_+ \cap V(G^\circ) \setminus \Big(V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \bigcup \mathcal{L}^\circ\Big) \subseteq V(G_{\mathrm{exp}}) \cup (V_{\rightsquigarrow\mathbb{E}} \cap \mathbb{L}_{\eta,k}(G)) \,. \tag{6.41}$$

Indeed, observe that by (3.8),

$$\begin{aligned} V_+ \cap V(G^\circ) &\subseteq V(\mathcal{M}_A \cup \mathcal{M}_B) \cup V(G_{\mathrm{exp}}) \cup \big(\mathbb{L}_{\eta,k}(G) \setminus (L_\# \cup V_{\rightsquigarrow\mathbb{H}} \cup \mathbb{J}_1)\big) \\ &\subseteq V(\mathcal{M}_A \cup \mathcal{M}_B) \cup V(G_{\mathrm{exp}}) \cup \big(\mathbb{L}_{\frac{9\eta}{10},k}(G_\nabla) \setminus (V_{\rightsquigarrow\mathbb{H}} \cup \mathbb{J}_1)\big) \,. \end{aligned}$$

So, in order to show (6.41), it suffices to see that for each $X \in \mathcal{V}^\circ \setminus \mathcal{V}(\mathcal{M}_A \cup \mathcal{M}_B)$ with $X \subseteq \mathbb{L}_{\frac{9\eta}{10},k}(G_\nabla) \setminus (V_{\rightsquigarrow\mathbb{H}} \cup \mathbb{J}_1 \cup V(G_{\mathrm{exp}}) \cup V_{\rightsquigarrow\mathbb{E}})$, we have $X \in \mathcal{L}^\circ$. So assume X

is as above. Let $v \in X$. We calculate

$$\begin{aligned}
\deg_{G_{\mathrm{reg}}}(v, V(G^\circ)) &\geqslant \deg_{G_{\mathrm{reg}}}(v, V(\mathcal{M}_A \cup \mathcal{M}_B)) \\
{\scriptstyle (v \notin V(G_{\mathrm{exp}}))} \quad &\geqslant \left(1 + \frac{9\eta}{10}\right) k - \deg_G(v, \mathbb{H}) - \deg_{G_{\mathcal{D}}}(v, \mathbb{E}) \\
&\quad - \deg_{G_{\mathrm{reg}}}\left(v, \bigcup \mathbf{V} \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)\right) \\
{\scriptstyle (v \notin V_{\rightsquigarrow \mathbb{H}} \cup V_{\rightsquigarrow \mathbb{E}} \cup \mathbb{J}_1 \cup V(\mathcal{M}_A \cup \mathcal{M}_B))} \quad &\geqslant \left(1 + \frac{9\eta}{10}\right) k - \frac{\eta k}{100} - \frac{\rho k}{100\Omega^*} - \gamma k \\
&\geqslant \left(1 + \frac{\eta}{20}\right) k\ .
\end{aligned}$$

We deduce that $X \in \mathcal{L}^\circ$, completing the proof of (6.41).

Next, observe that by the definition of $\mathcal{C}$, we have

$$\begin{aligned}
V_+ \cap V(G^\circ) &\supseteq V_{\mathrm{good}} \cap V(G^\circ) \\
&\supseteq V_{\mathrm{good}} \setminus \big(V_{\mathrm{good}} \setminus V(G^\circ)\big) \\
&\supseteq V_{\mathrm{good}} \setminus \Big(V_{\rightsquigarrow \mathbb{H}} \cup \mathbb{J}_1 \cup \bigcup \mathcal{C}^- \cup \mathbb{E} \cup V(G_{\mathrm{exp}})\Big). \qquad (6.42)
\end{aligned}$$

We are now ready to prove Subclaim 6.13.1.1. For each vertex $v \in A \setminus S$, we have

$$\begin{aligned}
\deg_{G_{\mathrm{reg}}}\left(v, V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \bigcup \mathcal{L}^\circ\right) &\geqslant \deg_{G_{\mathrm{reg}}}(v, V_+ \cap V(G^\circ)) \\
&\quad - \deg_{G_{\mathrm{reg}}}\left(v, (V_+ \cap V(G^\circ)) \setminus (V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \mathcal{L}^\circ)\right) \\
{\scriptstyle (\text{by } (6.42),\ (6.41))} \quad &\geqslant \deg_{G_{\mathrm{reg}}}(v, V_{\mathrm{good}}) - \deg_{G_{\mathrm{reg}}}\left(v, V_{\rightsquigarrow \mathbb{H}} \cup \mathbb{J}_1 \cup \bigcup \mathcal{C}^-\right) \\
&\quad - \deg_{G_{\mathrm{reg}}}(v, \mathbb{E}) - 2\deg_{G_{\mathrm{reg}}}(v, V(G_{\mathrm{exp}})) \\
&\quad - \deg_{G_{\mathrm{reg}}}\left(v, (V_{\rightsquigarrow \mathbb{E}} \cap \mathbb{L}_{\eta,k}(G)) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)\right) \\
{\scriptstyle (\text{by } (6.38),\text{ as } v \notin S \cup \mathbb{J},\text{ by } (\mathbf{t5}))} \quad &\geqslant \left(1 + \frac{\eta}{20}\right) k - \frac{\eta^2 k}{10^5} - \frac{\eta k}{200} - \frac{\rho k}{100\Omega^*} - 2\rho k - \frac{2\eta^2 k}{10^5} \\
&> \left(1 + \frac{\eta}{40}\right) k + \frac{\eta k}{200}\ ,
\end{aligned}$$

where for the second-to-last inequality we used the abbreviation "by (**t5**)" to indicate that this case implies that

$$v \notin \mathrm{shadow}_{G_\nabla}(V(G_{\mathrm{exp}}), \rho k) \cup \mathrm{shadow}_{G_\nabla}\left((V_{\rightsquigarrow \mathbb{E}} \cap \mathbb{L}_{\eta,k}(G)) \setminus V(\mathcal{M}_A \cup M_B), \frac{2\eta^2 k}{10^5}\right).$$

As $|A \setminus S| \geqslant \frac{|A|}{2}$, we note that the set A satisfies the requirements of the claim.

The same calculations hold for the set B. This finishes the proof of Subclaim 6.13.1.1. □

The next auxiliary subclaim is needed in our proof of Claim 6.13.1 in case (**M2**).

SUBCLAIM 6.13.1.2. *Suppose that case* (**M2**) *occurs. Then there exists an edge* $C_A C_B \in E(\mathbf{G}_{\mathrm{reg}})$ *such that* $\deg_{G_{\mathrm{reg}}}(v, V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \bigcup \mathcal{L}^\circ) \geqslant (1 + \frac{\eta}{40})k + \frac{\eta k}{400}$ *for all but at most* $2\varepsilon' \mathfrak{c}$ *vertices* $v \in C_A$, *and all but at most* $2\varepsilon' \mathfrak{c}$ *vertices* $v \in C_B$. *Moreover, there exist* $A, B \in \mathcal{V}(\mathcal{M})$ *such that* $|C_A \cap A| > \sqrt{\varepsilon'}\mathfrak{c}$ *and* $|C_B \cap B| > \sqrt{\varepsilon'}\mathfrak{c}$.

Proof of Subclaim 6.13.1.2. Let $(A,B)\in\mathcal{M}$ be given as in Subclaim 6.13.1.1. Let $P_A\subseteq A$ and $P_B\subseteq B$ be the vertices which fail the assertion of Subclaim 6.13.1.1. Note that with this notation, Subclaim 6.13.1.1 states that

$$|A\setminus P_A|\geqslant |A|/2\,. \tag{6.43}$$

Call a cluster $C\in\mathbf{V}$ *A-negligible* if $|C\cap(A\setminus P_A)|\leqslant\frac{\gamma^3\mathfrak{c}}{16\Omega^*k}|A|$. Let R_A be the union of all A-negligible clusters.

Recall that (A,B) is entirely contained in one dense spot from $(U,W;F)\in\mathcal{D}_\nabla$ (cf. (**M2**)). So by Fact 2.7, and since the spots in $\mathcal{D}_\nabla$ are $(\frac{\gamma^3k}{4},\frac{\gamma^3k}{4})$-dense, we know that $\max\{|U|,|W|\}\leqslant\frac{4\Omega^*k}{\gamma^3}$. In particular, there are at most $\frac{4\Omega^*k}{\gamma^3\mathfrak{c}}$ A-negligible clusters which intersect $A\cap R_A$.

As these clusters are all disjoint, we find that

$$|(A\cap R_A)\setminus P_A|\leqslant\frac{4\Omega^*k}{\gamma^3\mathfrak{c}}\cdot|C\cap(A\setminus P_A)|\leqslant\frac{|A|}{4}\,.$$

This gives

$$|A\setminus(P_A\cup R_A)|\overset{(6.43)}{\geqslant}\frac{|A|}{2}-|(A\cap R_A)\setminus P_A|\geqslant\frac{|A|}{4}\,.$$

Similarly, we can introduce the notion of B-negligible clusters and the set R_B, and get $|(B\cap R_B)\setminus P_B|\leqslant\frac{|B|}{4}$ and $|B\setminus(P_B\cup R_B)|\geqslant\frac{|B|}{4}$.

By the regularity of the pair (A,B) there exists at least one edge

$$ab\in E(G^*[A\setminus(P_A\cup R_A),B\setminus(P_B\cup R_B)])\,,$$

where $a\in A$, $b\in B$, and G^* is the graph formed by the edges of $\mathcal{D}_\nabla$. As $V(\mathcal{M})\subseteq V(G_{\mathrm{reg}})$ by the assumption of case (**t5**), we have that $ab\in E(G_{\mathrm{reg}})$. Let $C_A,C_B\in\mathbf{V}$ be the clusters containing a and b, respectively. Note that $C_AC_B\in E(\mathbf{G}_{\mathrm{reg}})$.

Now as $a\notin R_A$, also C_A is disjoint from R_A, and thus

$$|C_A\cap(A\setminus P_A)|>\frac{\gamma^3\mathfrak{c}}{16\Omega^*k}\cdot\frac{\widehat{\alpha}\rho k}{\Omega^*}>\sqrt{\varepsilon'}\mathfrak{c}\,.$$

This proves the "moreover" part of the claim for C_A. So there are at least $2\varepsilon'\mathfrak{c}$ vertices v in C_A with $\deg_{G_{\mathrm{reg}}}(v,V(\mathcal{M}_A\cup\mathcal{M}_B)\cup\bigcup\mathcal{L}^\circ)\geqslant(1+\frac{\eta}{40})k+\frac{\eta k}{200}$ (by the definition of P_A). By Lemma 2.3, and using Facts 2.7 and 2.8, we thus have that $\deg_{G_{\mathrm{reg}}}(v,V(\mathcal{M}_A\cup\mathcal{M}_B)\cup\bigcup\mathcal{L}^\circ)\geqslant(1+\frac{\eta}{40})k+\frac{\eta k}{400}$ for all but at most $2\varepsilon'\mathfrak{c}$ vertices v of C_A. The same calculations hold for C_B. ◻

In the remainder of the proof of Claim 6.13.1 we have to distinguish between cases (**M1**) and (**M2**).

Let us first consider the case (**M2**). Let $C_A,C_B\in\mathbf{V}$ and $A,B\in\mathcal{V}(\mathcal{M})$ be given by Subclaim 6.13.1.2. We have $|C_A\setminus(V_{\rightsquigarrow\mathbb{H}}\cup L_\#\cup\mathbb{J}_1)|>\sqrt{\varepsilon'}|C_A|$ by Subclaim 6.13.1.2 and by the definition of $\mathcal{M}$ and the definition of $\mathbb{J}$. Thus, $C_A\cap V(G^\circ)$ is nonempty. Let $X_A\in\mathcal{V}^\circ$ be an arbitrary set in C_A. Similarly, we obtain a set $X_B\in\mathcal{V}^\circ$, $X_B\subseteq C_B$. The claimed properties of the pair (X_A,X_B) follow directly from Subclaim 6.13.1.2.

It remains to treat the case (**M1**). Let (A,B) be from Subclaim 6.13.1.1. Let $(X_A,X_B)\in\mathcal{M}_{\mathrm{good}}$ be such that $X_A\supseteq A$ and $X_B\supseteq B$. Claim 6.13.1.1 asserts that at least

$$\frac{|A|}{2}\overset{(\mathbf{M1})}{\geqslant}\frac{\eta^2\mathfrak{c}}{2\cdot 10^4}>2\varepsilon'\mathfrak{c}$$

vertices of A have large degree (in G_{reg}) into the set $V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \bigcup \mathcal{L}^{\circ}$. Therefore, by Lemma 2.3, X_A and X_B satisfy the assertion of the claim.

This proves Claim 6.13.1. □

Recall that Claim 6.13.1 was the only missing piece in the proof of Lemma 6.13. The proof of Lemma 6.13 is thus complete. □

The proof of Lemma 6.3 follows by putting together Lemmas 6.8, 6.9, 6.11, 6.12, and 6.13.

Acknowledgments. The work on this project took place from the beginning of 2008 until 2014, and the authors are very grateful to the following institutions and funding bodies for their support.

During the work on this paper Hladký was also affiliated with Zentrum Mathematik, TU Munich, and the Department of Computer Science, University of Warwick. Hladký was funded by a BAYHOST fellowship, a DAAD fellowship, Charles University grant GAUK 202-10/258009, EPSRC award EP/D063191/1, and an EPSRC Postdoctoral Fellowship during the work on the project.

Komlós and Szemerédi acknowledge the support of NSF grant DMS-0902241.

Piguet was also affiliated with the Institute of Theoretical Computer Science, Charles University in Prague; Zentrum Mathematik, TU Munich; the Department of Computer Science and DIMAP, University of Warwick; and the School of Mathematics, University of Birmingham. The work leading to this invention was supported by the European Regional Development Fund (ERDF), project "NTIS–New Technologies for the Information Society," European Centre of Excellence, CZ.1.05/1.1.00/02.0090. Piguet acknowledges the support of the Marie Curie fellowship FIST, DFG grant TA 309/2-1, a DAAD fellowship, Czech Ministry of Education project 1M0545, EPSRC award EP/D063191/1, and the EPSRC Additional Sponsorship, with a grant reference of EP/J501414/1, which allowed her to travel with her young child and thus continue to collaborate closely with her coauthors on this project. This grant was also used to host Stein in Birmingham.

Stein was affiliated with the Institute of Mathematics and Statistics, University of São Paulo, and the Centre for Mathematical Modeling, University of Chile. She was supported by an FAPESP fellowship, by FAPESP travel grant PQ-EX 2008/50338-0, and also by CMM-Basal and FONDECYT grants 11090141 and 1140766. She also received funding by EPSRC Additional Sponsorship EP/J501414/1.

The authors enjoyed the hospitality of the School of Mathematics of the University of Birmingham; the Center for Mathematical Modeling, University of Chile; the Alfréd Rényi Institute of Mathematics of the Hungarian Academy of Sciences; and Charles University in Prague during their long-term visits.

The yet unpublished work of Ajtai, Komlós, Simonovits, and Szemerédi on the Erdős–Sós conjecture was the starting point for the authors' project, and their solution crucially relies on the methods developed for the Erdős-Sós conjecture. Hladký, Piguet, and Stein are very grateful to the former group for explaining those techniques.

A doctoral thesis titled *Structural Graph Theory* submitted by Hladký in 2012 under the supervision of Daniel Král at Charles University in Prague is based on the series of papers [HKP+a, HKP+b, HKP+c, HKP+d]. The texts of the two works overlap greatly. The authors are grateful to Ph.D. committee members Peter Keevash and Michael Krivelevich. Their valuable comments are reflected in the series.

The authors thank the referees for their very detailed remarks.

The contents of this publication reflects only the authors' views and not necessarily the views of the European Commission of the European Union.

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