On the Multi-coloured Ramsey Numbers of Cycles

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Abstract

For a graph L and an integer $k \geq 2$, $R_k(L)$ denotes the smallest integer N for which for any edge-colouring of the complete graph K_N by k colours there exists a colour i for which the corresponding colour class contains L as a subgraph.

Bondy and Erdős conjectured that for an odd cycle C_n on n vertices,

$$R_k(C_n) = 2^{k-1}(n-1)+1 \quad \text{for } n > 3.$$

They proved the case when $k = 2$ and also provided an upper bound $R_k(C_n) \leq (k+2)!n$. Recently, this conjecture has been verified for $k = 3$ if n is large. In this note, we prove that for every integer $k \geq 4$,

$$R_k(C_n) \leq k2^k n + o(n), \quad \text{as } n \to \infty.$$

When n is even, Yongqi, Yuansheng, Feng, and Bingxi gave a construction, showing that $R_k(C_n) \geq (k-1)n - 2k + 4$. Here we prove that if n is even, then

$$R_k(C_n) \leq kn + o(n), \quad \text{as } n \to \infty.$$

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1. Introduction

In this note we shall consider Ramsey problems connected to edge-colourings of ordinary graphs with k colours, for a given $k \geq 2$, and try to ensure monochromatic cycles of a given length. We shall use the standard notation. Given a graph $G = (V, E)$, $v(G)$ denotes the number of vertices and $e(G)$ the number of edges in G. For a subset W of V, $G[W]$ is the subgraph of G induced by the vertices in W.

For graphs $L_1, \ldots, L_k$, the Ramsey number $R(L_1, \ldots, L_k)$ is the minimum integer N such that for any edge-colouring of the complete graph K_N by k colours there exists a colour i for which the i^{th} colour class contains L_i as a subgraph. For $L_1 = L_2 = \ldots = L_k = L$, we set $R_k(L) := R(L_1, \ldots, L_k)$.

The behaviour of Ramsey number $R(C_n, C_m)$ has been studied by several authors, for example, Bondy and Erdős, [2], Faudree and Schelp, [4], Rosta, [10], and it is completely described and well-understood. Among others, it is known that

$$R_2(C_n) = \begin{cases} 2n - 1, & \text{if } n \geq 5 \text{ is odd,} \\ \frac{3n}{2} - 1, & \text{if } n \geq 6 \text{ is even.} \end{cases}$$

Bondy and Erdős [2] conjectured that $R_k(C_n) = 2^{k-1}(n-1)+1$ for odd $n > 3$. The conjectured extremal colouring, giving the lower bound, can be easily constructed recursively: for two colours, take two disjoint sets of size $n - 1$ and colour all pairs within each set by colour 1 and all pairs joining them by colour 2. For $i = 3, \ldots, k$, take two disjoint copies of the colouring for $i - 1$ colours and colour all pairs joining these two copies by colour i. The final k-colouring has $2^{k-1}(n - 1)$ vertices and every monochromatic component has either only $n - 1$ vertices or it is bipartite and therefore does not contain odd cycles.

As for the upper bound for $R_k(C_n)$, $k \geq 3$, Łuczak [9] proved that if n is odd, then $R_3(C_n) = 4n + o(n)$, as $n \to \infty$. Later, Kohayakawa, Simonovits, and Skokan [7, 8] showed that $R_3(C_n) = 4n - 3$ for all odd, sufficiently large values of n. The conjecture is still open for $k \geq 4$. Bondy and Erdős [2] remarked that they could prove $R_k(C_n) \leq (k + 2)!n$ for n odd. In this note we shall give an upper bound which is correct up to $O(k)$ factor.

Theorem 1. *For every $k \geq 4$ and odd n,*

$$R_k(C_n) \leq k2^k n + o(n), \qquad \text{as } n \to \infty.$$

The Ramsey number $R_k(C_n)$ behaves rather differently for even values of n. From [4] and [10], we know that $R_2(C_n) = 3n/2 - 1$ and, for large even n, Benevides and Skokan [1] proved that $R_3(C_n) = 2n$. Yongqi, Yuansheng, Feng, and Bingxi [11] gave a construction yielding

$$R_k(C_n) \geq (k-1)n - 2k + 4.$$

Here we prove the following.

Theorem 2. *For every $k \geq 2$ and even n,*

$$R_k(C_n) \leq kn + o(n), \qquad \text{as } n \to \infty.$$

The difference between the lower and upper bounds is only $n + o(n)$ and we think that the lower bound is sharp.

2. Tools

We shall make use of the following result of Erdős and Gallai, [3].

Theorem 3. *Let $n \geq 3$. For any graph G with at least $(n-1)(v(G)-1)/2+1$ edges, G contains a cycle of length at least n.*

The next lemma of Figaj and Łuczak ([5], Lemma 9) describes some structural properties of graphs without long odd cycles.

Lemma 4. *If no non-bipartite component of a graph G contains a matching of at least $n/2$ edges, then there exists a partition $V(G) = V^1 \cup V^2 \cup V^3$ of the vertices of G for which*

- *(A) G has no edges joining $V^1 \cup V^2$ and V^3;*
- *(B) the subgraph $G[V^1 \cup V^2]$ is bipartite, with bipartition (V^1, V^2);*
- *(C) the subgraph $G[V^3]$ has at most $n(|V^3|-1)/2$ edges and each component of $G[V^3]$ is non-bipartite.*

Notice that Lemma 4 defines a decomposition of $V(G)$ into sets V^1, V^2, and V^3, and we shall call V^3 the *sparse set*.

3. Odd cycles

Our proof of Theorem 1 is based on the following lemma of Figaj and Łuczak; see Lemma 3 in [6] for a more general statement.

Lemma 5. *Let a real number $c > 0$ be given. If for every $\varepsilon > 0$ there exist a $\delta > 0$ and an n_0 such that for every odd $n > n_0$ and any graph G with $v(G) > (1+\varepsilon)cn$ and $e(G) \geq (1-\delta)\binom{v(G)}{2}$, any k-edge-colouring of G has a monochromatic non-bipartite component with a matching of $(n+1)/2$ edges, then*

$$R_k(C_n) \leq (c + o(1))n, \qquad \text{as } n \to \infty.$$

Hence, Theorem 1 follows from the next lemma.

Lemma 6. *Given a natural number $k \geq 4$ and an $\varepsilon > 0$, let n be a sufficiently large odd integer, $\delta = \varepsilon/2^{2k+4}$ and $N = (1+\varepsilon)k2^k n$. Suppose that G is a graph with $v(G) \geq N$ and $e(G) \geq (1-\delta)\binom{v(G)}{2}$. Then in any k-colouring of the edges of G, there exists a monochromatic non-bipartite component containing a matching of $(n+1)/2$ edges.*

Proof. Assume to the contrary that there exists a k-edge colouring of G without a monochromatic matching of $(n+1)/2$ edges in a non-bipartite component. We may also assume that $\varepsilon < 1$ and $v(G) = N$. Indeed, if $v(G) > N$ and

$$e(G) \geq (1-\delta)\binom{v(G)}{2}, \tag{1}$$

then, iteratively removing ($v(G) - N$ times) a vertex of minimum degree, we obtain a subgraph of G with N vertices and at least $(1-\delta)\binom{N}{2}$ edges.

For every colour i, let G_i be the spanning subgraph of G induced by the edges coloured by i. Then no G_i contains a matching of $(n+1)/2$ edges in a non-bipartite component, otherwise G_i would satisfy the conclusion of the lemma.

We apply Lemma 4 to G_i for every $i \in [k] := \{1, \ldots, k\}$ and obtain a partition into V_i^1, V_i^2, and the sparse set V_i^3. For every $i \in [k]$, set $X_i^1 = V_i^1$ and $X_i^2 = V_i^2 \cup V_i^3$. Notice there are 2^k sets of the form $\bigcap_{\ell=1}^{k} X_\ell^{j_\ell}$, where $j_\ell \in \{1, 2\}$ for every ℓ. Since V_i^1, V_i^2 and V_i^3 is a partition of $V(G)$ for every i, it is clear that these sets are pairwise disjoint and form a partition of $V(G)$.

The graph G has $N = (1+\varepsilon)k2^k n$ vertices, therefore, there is a choice of $j_\ell \in \{1,2\}$, $\ell = 1, 2, \ldots, k$, such that the size of the set $X = \bigcap_{\ell=1}^{k} X_\ell^{j_\ell}$ is at least $N/2^k = (1+\epsilon)kn > kn$.

For every i, if there is and edge e of colour i in X, then it must be contained in V_i^3 (by (A) and (B)). Hence, it is contained in an odd component (by (C)). Since there is no monochromatic matching of $(n+1)/2$ edges in a non-bipartite component, X contains no cycles longer than n in colour i, so, by Theorem 3, there are at most $n(|X|-1)/2$ edges of colour i with both endpoints in X. Hence,

$$e(G[X]) \leq kn(|X|-1)/2. \tag{2}$$

On the other hand, from (1), we have

$$e(G[X]) \geq \binom{|X|}{2} - \delta\binom{N}{2}. \tag{3}$$

Comparing (2) and (3) yields

$$|X| \leq kn + \delta\frac{N(N-1)}{|X|-1}.$$

Using assumptions $\varepsilon < 1$, $\delta = \varepsilon/2^{2k+4}$, $N \leq k2^{k+1}$, and $|X| > kn$, we have that

$$\delta\frac{N(N-1)}{|X|-1} \leq 2\delta\frac{N^2}{|X|} \leq 2\delta\frac{(k2^{k+1})^2}{kn} \leq \frac{\varepsilon kn}{2}.$$

Thus,

$$(1+\varepsilon)kn \leq |X| \leq kn + \frac{\varepsilon kn}{2},$$

which is a contradiction. ∎

Remark 7. The methods of Figaj and Łuczak and the proof above give a slightly stronger result than Theorem 1.

Given a natural number $k \geq 4$ and an $\varepsilon > 0$, there exist a $\delta > 0$ and an n_0 with the following property. Suppose that $n > n_0$ is odd, $N \geq (1+\varepsilon)k2^k n$, and G is a graph with $v(G) \geq N$ and $e(G) \geq (1-\delta)\binom{v(G)}{2}$. Then in any k-colouring of the edges of G, there exists a monochromatic cycle C_n.

These types of theorems are not much more difficult than the ones on the colourings of the complete graphs, however, these are the forms we use in our applications.

4. Even cycles

In the proof of Theorem 2 we shall use another case of the lemma of Figaj and Łuczak (Lemma 3 in [6]).

Lemma 8. *Let a real number $c > 0$ be given. If for every $\varepsilon > 0$ there exist a $\delta > 0$ and an n_0 such that for every even $n > n_0$ and any graph G with $v(G) > (1+\varepsilon)cn$ and $e(G) \geq (1-\delta)\binom{v(G)}{2}$, any k-edge-colouring of G has a monochromatic component containing a matching of $n/2$ edges, then*

$$R_k(C_n) \leq (c + o(1))n.$$

Now we prove Theorem 2.

Proof. For an arbitrary $0 < \varepsilon < 1$, consider any k-colouring of a graph G on $N > (1+\varepsilon)nk$ vertices and with at least $(1-\varepsilon/3)\binom{N}{2}$ edges. One of the colours must have at least $\frac{1}{k}(1-\varepsilon/3)\binom{N}{2} > \frac{1}{2}n(N-1)+1$ edges, so, by Theorem 3, this colour contains a cycle of length at least $n+1$. This implies the existence of a matching covering n vertices in a monochromatic component. Hence, Lemma 8 implies that $R_k(C_n) \leq (k+o(1))n$. ∎

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