NOTE ON A HYPERGRAPH EXTREMAL PROBLEM

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Introduction. We shall consider 3-uniform hypergraphs "without loops or multiple edges." This means that we shall consider a set X which will be called the vertex-set of the hypergraph and a set of unordered triples from X called the triples of the hypergraph. The expression "without loops" means that each triple has 3 different elements and the expression "without multiple edges" means that each triple can occur at most once in the set of edges.

Problem. Let L be a family of hypergraphs. What is the maximum number of triples a hypergraph on n vertices can have if it does not contain a subhypergraph isomorphic to some members of L ? For a given finite or infinite family L , the problem asked above will be called an extremal problem; the maximum will be denoted by ext(n ; L) ; the members of L will be called sample hypergraphs, and the hypergraphs attaining the maximum generally are called extremal hypergraphs.

Definition. The r-pyramid $L_{r,t}$ based on a polygon of t vertices is the hypergraph defined on the r + t vertices $x_1, \dots, x_r ; y_1, \dots, y_{t-1}, y_t, y_{t+1} = y_1$ and having the triples

$$(x_i , y_j , y_{j+1}) \quad (i=1, \dots, r ; j=1, \dots, t)$$

Further, L_r is the family of all the r-pyramids $L_{r,t}$, $t = 2, 3, \dots$.

Theorem. $\text{ext}(n ; L_3) > (\frac{1}{6} + o(1))\, n^{8/3}$.

(This means that there are hypergraphs with n vertices and almost $\frac{1}{6} n^{8/3}$ triples which do not contain any 3-pyramid.)

Remarks. 1) In [1] we needed and proved the following lemma:

$$\text{ext}(n ; L_r) = O(n^{3 - \frac{1}{r}}) , \quad r = 1, 2, 3, \dots \tag{1}$$

This lemma was needed to prove that a 4-colour-critical hypergraph cannot have too many independent vertices. As a matter of fact, we needed this result only for r = 2 . At the same time W. Brown, P. Erdos and V. T. Sos, [1] among some other hypergraph extremal theorems proved that if T is the family of sample hypergraphs obtained from the triangulations of the 3-sphere, then

(2) $$\mathrm{ext}(n\ ;\ T) = O(n^{3 - \frac{1}{2}}) \quad .$$

They used the fact that T contains L_2, proved (1) for $r = 2$, from which (2) followed trivially. Using a finite geometrical construction, they also proved that (2) is sharp, i.e. the exponent is the best possible. We used a so-called probabilistic argument to prove the weaker assertion that (1) is sharp for $r = 2$. The main purpose of this paper is to prove that (1) is sharp for $r = 3$ as well.

2) There is a result of Kovary, Turan and T. Sos [3] asserting that if $K_2(p\ ,\ q)$ is the complete bipartite graph with p and q vertices in its first and second classes respectively, then

(3) $$\mathrm{ext}(n\ ;\ K_2(p\ ,\ q)\ ) = O(n^{2 - \frac{1}{p}})\ , \quad (p \leq q)$$

It can be conjectured that this result is the best possible; however, this is not proved except for $p = 1$, 2, and 3. If we can prove that (1) is sharp for $r = 4$, then it will follow that (3) is also sharp for $p = 4$. This suggests that it will be difficult to prove the sharpness of (1) for $r = 4$.

The construction. The construction given below will be based on the construction of W. G. Brown [4], showing that (3) is sharp for $p = 3$. In view of the second remark, this is not "surprising" at all. First we define the graph of Brown. Let p be an odd prime and the vertices of our graph be the points in the 3-dimensional affine space over the field $GF(p)$, i.e. over the field of residues mod p.

Let us join two points $\underline{x}$ and $\underline{y}$ by an edge if

(4) $$\sum_{i=1}^{3} (x_i - y_i)^2 = a$$

where a is a quadratic residue or non-residue depending on whether p has the form $4s + 3$ or $4s + 1$, $a \neq 0$ and is constant for a given graph. According to a well-known theorem of Lebesgue [5, p. 325] the valences of each vertex will be $p^2 - p$ and, as Brown proves, this graph never contains a $K_2(3,3)$. Let us join 3 points $\underline{x}$, $\underline{y}$, $\underline{z}$ in the graph of Brown by a triple if

(5) $$\sum_{i=1}^{3} (x_i + y_i + z_i)^2 = a .$$

This hypergraph contains some 3-pyramids of very special "position". We omit a few triples and prove that the obtained hypergraph does not contain 3-pyramids. Let us denote the graph of Brown by B , the hypergraph defined above by A and let U be the hypergraph obtained from A by omitting all those vertices $\underline{x} = (x_1, x_2, x_3)$ for which

(6) at least one of x_1 , x_2 , x_3 vanishes .

Of course we omit all the triples at least one vertex of which has already been omitted. Since in U each coordinate can be chosen in $p - 1$ different ways, U has

(7) $$(p - 1)^3$$

vertices. For every edge $(\underline{x} , \underline{c})$ of B , there are $\frac{p^3 - 3}{2}$ pairs $(\underline{y} , \underline{z})$ such that $\underline{y} + \underline{z} = \underline{c}$, $\underline{y} \neq \underline{z}$, $\underline{y} \neq \underline{x}$, $\underline{z} \neq \underline{x}$. These $(\underline{x} , \underline{y} , \underline{z})$ triples will belong to A and each triple of A can be counted only 3 times this way, so A has $\frac{p^8}{6} + O(p^7)$ triples and each vertex of A has the valence $\frac{p^5}{2} + O(p^4)$. Since we omitted only $O(p^2)$ vertices, the number of triples in U is

(8) $$\frac{p^8}{6} + O(p^7) .$$

We shall prove that U does not contain 3-pyramids.

Let us suppose that the vertices $\underline{x}_i$ $(i = 1 , 2 , 3)$ and $\underline{y}_j$ $(j = 1,\dots,k)$ define an $L_{3,k}$ in U . The triples of this $L_{3,k}$ are the triples $(\underline{x}_i, \underline{y}_j , \underline{y}_{j+1})$. Let $\underline{w}_j = -(\underline{y}_j + \underline{y}_{j+1})$. According to (4) and (5) the vertices $\underline{x}_i$ are joined to the vertices $\underline{w}_j$ in the graph B . We know that the vertices $\underline{x}_i$ are all different and that the graph B does not contain a complete bipartite graph with 3 vertices in each class. Hence there are at most 2 different vertices among $\underline{w}_1 ,\dots,\underline{w}_k$. On the other hand

(9) $$-(\underline{w}_j - \underline{w}_{j-1}) = \underline{y}_j + \underline{y}_{j+1} - \underline{y}_{j-1} - \underline{y}_j = \underline{y}_{j+1} - \underline{y}_{j-1} \neq 0 .$$

This gives that the set $\{\underline{w}_j : j = 1, \ldots, k\}$ has exactly 2 elements and every second element of it is the same. Therefore

(10) $$k = 2m \text{ and } \underline{w}_{2j} = \underline{u},\ w_{2j-1} = \underline{v} \quad (j = 1, \ldots, m),\ \underline{u} \neq \underline{v}.$$

Let us notice that from (9), (10) and $\underline{y}_{2m+1} = \underline{y}_1$ follows

$$0 = (\underline{y}_1 - \underline{y}_3) + (\underline{y}_3 - \underline{y}_5) + \ldots + (\underline{y}_{2m-3} - \underline{y}_{2m-1}) + (\underline{y}_{2m-1} - \underline{y}_{2m+1}) = m(\underline{u} - \underline{v})$$

and, since $\underline{u} \neq \underline{v}$, m must be a multiple of p. Until now we considered the larger hypergraph A. It is easy to see that A can contain 3-pyramids. However, U cannot contain these subgraphs. Indeed, the vertices $\underline{y}_{2i+1}$ form an arithmetic progression of vectors with increment $\underline{u} - \underline{v}$. Hence at least one coordinate, e.g. the first of $\underline{u} - \underline{v}$, is different from 0 ; therefore, the first coordinates of the vectors $\underline{y}_{2j+1}$ form an arithmetic progression of residues mod p. Since the number of elements in this progression is at least p, at least one of them must be 0 ; hence the corresponding $\underline{y}_{2j+1}$ does not belong to U. Thus, (7) and (8) complete the proof of our theorem if $n = (p-1)^3$. Since $\text{ext}(n\,;\,L)$ is monotone increasing in n and the primes are fairly dense among the integers (i.e. for every $\varepsilon > 0$ the interval $(n - \varepsilon n, n)$ contains a prime if n is large enough), our theorem follows for every n.

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