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Some of my Favorite Erdős Theorems and Related Results, Theories

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We shall survey here some parts of Paul Erdős's work and influence on graph theory, primarily through his influence on Extremal Graph Theory. There are many strongly related areas, above all, *Ramsey Theory* and *"Random Constructions"*. Then there are the theory of *Supersaturated Graphs,* the *Erdős–Kleitman–Rothschild theory,* the *Applications in Geometry* and *Applications in Number Theory,* and many further fields that were definitely among the favorite areas in Paul Erdős's graph theory. We shall describe the general theory of extremal graphs, and touch on some of the above subjects, sometimes only very shortly. I will only refer to those parts which are thoroughly described in some other papers of these volumes.

Beside these large areas there are many "isolated gems" in Erdős's combinatorics. Here we shall restrict ourselves only to a few ones.

1. Preface

The aim of these two volumes of survey articles is to provide a good overview of Paul Erdős's mathematics (and also of his personality). To learn about *his personality,* I would *not* recommend the books written about him. More adequate descriptions of Paul's personality and his life are the articles of Babai [13], [14], of T. Sós [262], [263] or Bollobás [35], or the paper by T. Sós and me, [258].[1] And, of course, one should read the "birthday" papers of

[1] . . . and there are many-many others, like that of Gy. Szekeres [271]. . . If one searches the MathSciNet for Paul Erdős and 1913, then one finds further ones, like [2, 15] [14], [18], and many others.

his old friends, like Paul Turán [281], or of Richard Rado [236] and Ernst Straus [268][2], written on the occasion of his 70th birthday and of Hajnal and T. Sós [170] or Bollobás [34].

To learn about Paul Erdős's *mathematics,* I feel, the best is to *read his original papers.* One of my favorite mathematics books is the Art of Counting, Paul's selected papers in Combinatorics [95]. I also warmly recommend Paul Turán's birthday paper written on the fiftieth birthday of Paul, reprinted here [281], or Paul's paper on his favorite theorems [103] or Bollobás' paper [35].

Here I decided to describe Paul Erdős's influence on Extremal Graph Theory "through my eyes", but not trying to give a very even description. The task I undertook is a hopeless one. András Hajnal, in his survey on Erdős's Set Theory [172] writes:

> "It is as if I have been trying to sketch a rain forest, but with only enough time and ability to draw the trunks of what I thought to be the largest trees. Paul's real strength is in the variety of some of those hundreds of small questions which he has asked that have given some real insights into so many different topics. I can only admire his inventiveness and thank him for everything he has given us."

When Paul became 80, as Hajnal and many others of us, I also wrote a survey, on Paul's influence on Extremal Graph Theory [255]. Of course, I cannot avoid some repetitions here: Erdős died when he was 83, and the facts did not change that much in Paul's last 2-3 years. Yet I shift the emphasis here to those parts which I had to neglect in [255] because of time, space and energy limitations, or where new results have been obtained recently. Occasionally I included older but more hidden ones.

Also, I tried to show his deep and wide influence on others. (At this point I should apologize: there are very many results and many people I should have mentioned here but could not, by "time, space and energy limitations".) Here I tried to avoid some of those parts which were covered by the papers of Bollobás [36], Bondy [48], Brown and myself (!) [60] and Spencer [266]. With a slight exaggeration, I could have used the title

Random walk in the Erdős jungle

I have not chosen this title: it is too fancy and large part of my rambling around in the "Erdős jungle" will not be that random.

[2] These two papers are reprinted here.

In the first major part I will describe the general theory, in the second one I will write about some areas more or less connected to the general theory.

There will be large parts of Erdős's graph theory which I do not intend to visit *now,* still I would like to point at them, describing them in just a few sentences, here or in the subsequent parts. These large parts are the following ones:

• *Erdős and Cycles:* described by me in [255] and in this volume by Bondy [48];

• *Multigraph and Digraph Extremal Problems:* A series of papers was written on the topic by Erdős, Brown and myself, described in this volume in the survey by Will Brown and me [60], see also our earlier paper on the "most general case" [59];

• *Ramsey–Turán Theory:* see the recent survey by V. T. Sós and me [259];

• *"Typical Structure of Random Graphs"*[3]*:* this will completely be avoided here. This very important and wide area is described in this volume by Bollobás [36]. At the same time I will occasionally touch on the application of random methods in graph theory.

• *Hypergraph Extremal Problems and Intersection Theorems:* one can identify them or distinguish between them. I regard them as two *distinct* subjects. I feel it is a pity that there are no separate and detailed surveys on them here. I will indicate only the topic of hypergraph extremal problems but not at the level it deserves. Of course, one could always read the surveys of Füredi [151], Frankl [145], or Füredi [152], and others.

• *Open Problems:* these constitute some of the most important ingredients of Erdős's influence. Again, the best sources are Erdős's own problem papers. I also strongly recommend [99]. A profound discussion of Erdős's open problems can be found in a paper of Fan Chung [72], or its expansion: the book of Fan Chung and R. L. Graham [73]. I will include here only very few of Erdős's open problems.

• *Szemerédi Regularity Lemma* [273]*:* This is one of the most powerful tools in Extremal Graph Theory. So it is one of the most powerful tools to solve many of the Erdős problems. Komlós and I have written an extensive survey on the topic, [200], and Komlós, Shokoufandeh, Szemerédi, and I [199] wrote a new version of it. In spite of the fact that many

[3] which we mostly call "The evolution of Random Graphs".

new and important results were proved in this field in the last few years, their description would be too technical here. Would I write on the topic, I would explain among others hypergraph versions, e.g., Frank, Rödl [144], F. Chung [70], some new variants due to Komlós, see, for example, [200], or some versions of Frieze and Kannan [148] and the fairly recent and very successful new technique, called **Blowup Lemma**, due to Komlós, G. N. Sárközy[4] and Szemerédi, see, e.g., [195], [196], [197], [198]. See also (among the many new, interesting results connected to the Regularity Lemma and the Blowup lemma) the survey of Komlós on the Blowup Lemma [194], Rödl and Ruciński [239], [240], etc.

- *Applications:* I will neither speak of the applications of Turán's theorem in the theory of distance-distributions, having applications in Geometry, Potential Theory, (initiated by Turán,[5] and developed into a theory by Erdős, Meir, T. Sós and Turán (see [282, 117], ...), nor about the applications in probability theory, initiated by G.O.H. Katona, see [186, 187], ...) since these are explained in [60] and also in [259]. We just touch on the applications in Number Theory.

Originally this survey was much longer. To cut it relatively short, I decided to leave out many parts here, but post a longer version of this paper (describing more topics, in more details) on my homepage:

`http://www.renyi.hu/~miki/erdos99.ps` .

It still shows a lot of compromises as to what to leave out and what to keep, but less than this version.

2. Introduction

Notation. We shall mostly consider simple graphs: graphs without loops and multiple edges. Yet, there will be parts where we shall regard digraphs and hypergraphs. For a set Q, $|Q|$ denotes its cardinality. Given a graph G, $e(G)$ denotes the number of its edges, $v(G)$ the number of vertices, $\chi(G)$ and $\alpha(G)$ its chromatic and independence numbers, respectively. For graphs the (first) subscript will mostly denote the number of vertices: G_n,

[4] In connection with cycles in graphs and extremal graph theory, we often refer to two Sárközy's: András, and his son, Gábor.

[5] actually emerging from an observation of Erdős.

$S_n, T_{n,p}, \dots$ denote graphs on n vertices. There will be some exceptions, e.g., speaking of excluded graphs $L_1, \dots, L_r$ we use subscripts just to enumerate them. Given two disjoint vertex sets, X and Y, in a graph G_n, $e(X,Y)$ denotes the number of edges joining X and Y. Given a graph G and a set X of vertices of G, the number of edges in the subgraph spanned by X will be denoted by $e(X)$, the subgraph of G spanned by X is $G[X]$. $G(X,Y)$ is a bipartite graph with color classes X and Y.

Special graphs. K_p denotes the complete graph on p vertices, $T_{n,p}$ is the Turán graph on n vertices and p classes: n vertices are partitioned into p classes as uniformly as possible and two vertices are joined iff they belong to different classes. This graph is the (unique) p-chromatic graph on n vertices with the maximum number of edges among such graphs. $K_p(n_1, \dots, n_p)$, often abbreviated to $K(n_1, \dots, n_p)$, denotes the complete p-partite graph with n_i vertices in its i^{th} class, $i = 1, 2, \dots, p$.

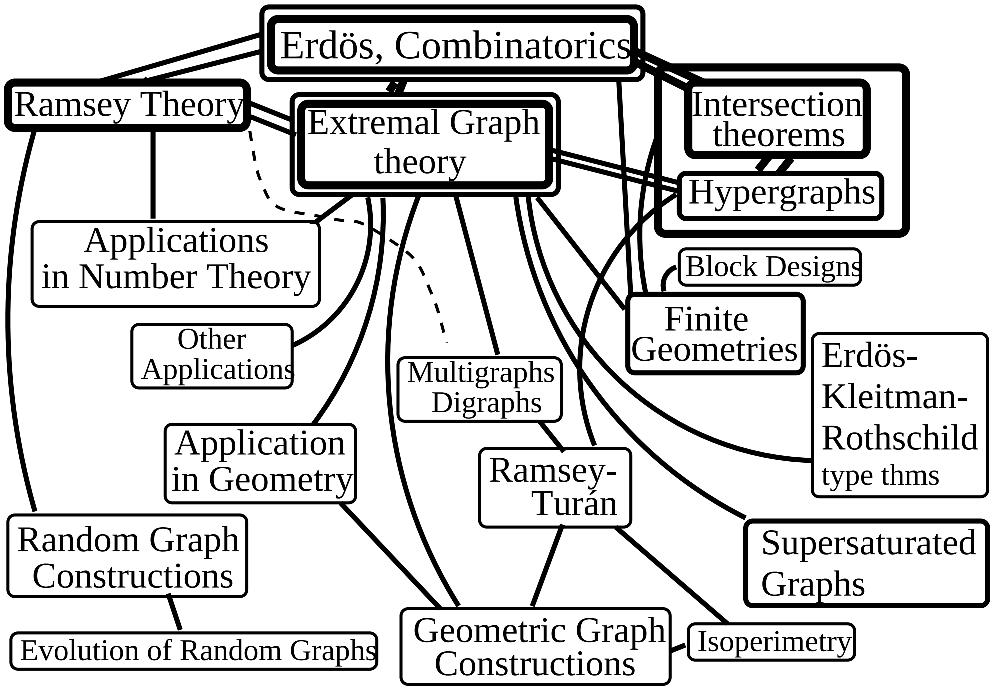
We shall say that X is **completely joined** to Y if every vertex of X is joined to every vertex of Y. Given two vertex-disjoint graphs, G and H, their **product** $G \otimes H$ is the graph obtained by joining each vertex of G to each one of H.[6] In case of many graphs we may also use the notation $\prod G_i$. As a generalization of $T_{n,p}$, we define $H_{n,p,d} := K_d \otimes T_{n-d,p}$.

Quoting. Below sometimes we quote some paragraphs from other papers, but the references and occasionally the notations too are changed to comply with ours.

2.1. A map?

Erdős's mathematics has an extremely wide scope. In Erdős's combinatorics the following four ingredients had very strong connections: Ramsey Theory, Extremal Graph Theory, Random Graphs, Applications in Number Theory and Geometry. Below I made a “map”, trying to describe this situation in a more detailed way. To make the picture below informative, I had to leave out several fields and many connections. The double lines indicate the most important connections.

[6] often called *join* of G and H.



2.2. Extremal Graph Theory: Early developments

Extremal graph theory is "one of the big gates to enter" the empire of Paul Erdős's combinatorics. We start with a brief non-mathematical description of the history and then go into details.

Formally, extremal graph theory started with Mantel's theorem [220] (1907)[7] which is the very special case of Turán's theorem, for K_3. Next Erdős proved and applied the C_4-theorem in number theory (1938), but missed the occasion to start the systematic investigation of this field. Then came around Turán's theorem (1941). The next step was that Erdős and A. H. Stone, setting out from a topological question, proved the Erdős–Stone theorem (1946) which later led to the Erdős–Stone–Simonovits theorem (1966) (yielding the general asymptotics in extremal graph problems) and to the Erdős–Simonovits structural theory (1967–68). Turán's theorem was rediscovered by A. Zykov [292], in 1949.

[7] Here I will always use the year of publication but often there are several years or between the birth and the publication of some results.

2.3. Starting from number theory

In his early years Erdős worked almost entirely in Number Theory. His papers between 1932 and 1936 were mostly about primes, and his first graph paper (a joint work with Tibor Gallai[8] and Endre Vázsonyi) appeared in Hungarian in 1936 and its topic was the Euler lines of infinite graphs [110].

Erdős often arrived at graph problems from applications in other fields. One famous case is the rediscovery of Ramsey's theorem in the Erdős–Szekeres "geometry" paper [137] (1935). A detailed description of this "story" can be found in the "preface"-paper of Szekeres in the Art of Counting [270].[9] Here we are interested in the birth of Extremal Graph Theory. In 1938 Erdős published a paper [80] with the title *"On sequences of integers no one of which divides the product of two others and related problems"*. This was where Erdős proved his first extremal graph theorem, to use it as a lemma, in a solution of a multiplicative Sidon problem:

Theorem 2.1. *If* $C_4 \not\subseteq G(X,Y)$, $|X| = |Y| = k$, *then* $e\big(G(X,Y)\big) \le 3k^{3/2}$.

The constant 3 can be improved to $1 + o(1)$ (see Section 4.7). Basically this theorem implied the upper bound in the main theorem of [80]. To get the lower bound Erdős used finite geometries (a construction of Eszter Klein).

Related literature: Erdős [93], T. Sós [261], Simonovits [255], Bondy [48] and Section 7.

3. Turán Type Extremal Problems

Extremal Graph Theory is one of the wider branches of Graph Theory and — in some sense — one of those where Paul Erdős's profound influence can really be seen and appreciated.

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[8] Many people changed their name to Hungarian-sounding ones. Gallai's name was originally Grünwald, Vázsonyi was originally called Weiszfeld.

[9] Turán wrote in [283] that "A decisive moment in his studies was the rediscovery of Ramsey's theorem in 1934."

We shall call the **Theory of Turán type Extremal Problems** the area which — though being much wider — still is originated from problems of the following type:

> Given a family $\mathcal{L}$ of sample graphs, what is the maximum number of edges a graph G_n can have without containing subgraphs from $\mathcal{L}$?

Here "subgraph" means "not necessarily induced". Given a family $\mathcal{L}$ of — so called — **excluded** or **forbidden** subgraphs, $\mathbf{ex}(n,\mathcal{L})$ denotes the maximum number of edges a graph G_n can have without containing forbidden subgraphs. (Again, containment does not assume "induced subgraph" of the given type.) The family of graphs attaining the maximum will be denoted by $\mathbf{EX}(n,\mathcal{L})$. If $\mathcal{L}$ consists of a single L, we shall use the notation $\mathbf{ex}(n,L)$ and $\mathbf{EX}(n,L)$ instead of $\mathbf{ex}\big(n,\{L\}\big)$ and $\mathbf{EX}\big(n,\{L\}\big)$. We call G $\mathcal{L}$-free if no $L\in\mathcal{L}$ is contained in G.

Perhaps Turán was the third to arrive at this field. In 1940 he proved the following theorem [278] (see also [279], [284]):

Theorem 3.1 (Turán). (a) *If G_n contains no K_p, then $e(G_n)\le e(T_{n,p-1})$.*

(b) *In case of equality $G_n = T_{n,p-1}$.*

Turán's original paper contains much more than just this theorem. Still, the main impact coming from Turán was that he asked the general question:

> *What happens if we replace K_p with some other forbidden graphs, e.g., with the graphs obtained from the Platonic polyhedra, or with a path of length ℓ, etc.*

3.1. Classification of ordinary extremal graph problems

When speaking of extremal graph problems, I basically distinguish three kinds of problems:

Non-degenerate problems: All the excluded graphs have chromatic number at least three. Therefore $\mathbf{ex}(n,\mathcal{L})\ge e(T_{n,2})\ge\left\lfloor\frac{n^2}{4}\right\rfloor$. The asymptotics are given by the Erdős–Stone–Simonovits theorem. Setting out from a problem in topology, Erdős and A. H. Stone proved the following theorem in 1946:

Theorem 3.2 (Erdős–Stone [135]). *For every fixed p and m*

$$\mathbf{ex}\big(n, K_{p+1}(m, \dots, m)\big) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + o(n^2). \tag{1}$$

Moreover, if p is fixed and $m := \sqrt{\ell_p(n)}$ where $\ell_p(x)$ denotes the p times iterated logarithm of x, (1) still holds.

Related literature: Bollobás and Erdős [37], Bollobás, Erdős, and Simonovits [38], Chvátal and Szemerédi [76, 77], Bollobás and Kohayakawa [42], Ishigami [177], and Lovász [212].

It turns out that a parameter related to the chromatic number plays a decisive role in many extremal graph theorems. The **subchromatic number** is defined by

$$p(\mathcal{L}) := \min\big\{\chi(L) \, : \, L \in \mathcal{L}\big\} - 1. \tag{2}$$

The following result is an easy consequence of the Erdős–Stone theorem [135]:

Theorem 3.3 (Erdős–Simonovits Theorem [125]). *If $\mathcal{L}$ is a family of graphs with subchromatic number p, then $\mathbf{ex}(n, \mathcal{L}) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + o(n^2)$.*

The meaning of this theorem is that $\mathbf{ex}(n, \mathcal{L})$ depends only very loosely on $\mathcal{L}$; up to an additive error term of order $o(n^2)$, it is already determined by the minimum chromatic number.[10] We shall return to the structural asymptotics in Section 3.2.

Degenerate problems: These are problems where there are bipartite excluded graphs in $\mathcal{L}$. We shall see in Section 4.1 that here $\mathbf{ex}(n, \mathcal{L}) = O(n^{2-c})$ for some $c = c(\mathcal{L}) > 0$. These seem to be very important problems from the point of view of understanding the general case, see Section 5. Among these, I feel, it is worth distinguishing the class of **very degenerate** extremal problems, where $\mathcal{L}$ contains some trees or forests, and therefore $\mathbf{ex}(n, \mathcal{L}) = O(n)$.

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[10] This does not assert to much if $\mathcal{L}$ contains bipartite graphs as well, see Section 4.

3.2. The general theory, structural results

The extremal problem for $\mathcal{L}$ is considered to be "completely solved" if all the extremal graphs have been found, at least for $n > n_0(\mathcal{L})$. Quite often this is too difficult, and we find only $\mathbf{ex}(n, \mathcal{L})$, or only good bounds for it. (These bounds may be asymptotically sharp, or sharp up to a multiplicative constant, or even weaker ones.)

We shall see several cases where the excluded graph is very complicated and yet we have a complete solution, e.g., the dodecahedron theorem, the icosahedron theorem, the Petersen graph theorem, see Section 6. In a slightly weaker sense the Octahedron theorem is also a "complete solution", (see Theorem 5.2).

The *asymptotical structure* of the extremal graphs is also almost determined by $p(\mathcal{L})$, for $p \geq 2$, and is very similar to that of $T_{n,p}$. This is expressed by the following results of Erdős and Simonovits [89, 91, 246]:

Theorem 3.4 (Asymptotic Structure Theorem). *Let $\mathcal{L}$ be a family of forbidden graphs with subchromatic number p. If S_n is any graph in $\mathbf{EX}(n, \mathcal{L})$, then it can be obtained from $T_{n,p}$ by deleting and adding $o(n^2)$ edges. Furthermore, if $\mathcal{L}$ is finite, then the minimum degree $d_{\min}(S_n) = \left(1 - \frac{1}{p}\right) n + o(n)$.*

The structure of extremal graphs is fairly *stable,* in the sense that the almost-extremal graphs have almost the same structure as the extremal graphs (for $\mathcal{L}$ or for K_{p+1}). This is expressed in our next result:

Theorem 3.5 (First Stability Theorem). *Let $\mathcal{L}$ be a family of forbidden graphs with subchromatic number $p \geq 2$. For every $\varepsilon > 0$, there exist a $\delta > 0$ and n_ε such that, if G_n contains no $L \in \mathcal{L}$, and if $e(G_n) > \mathbf{ex}(n, \mathcal{L}) - \delta n^2$, then, for $n > n_\varepsilon$, G_n can be obtained from $T_{n,p}$ by changing at most εn^2 edges.*

3.3. The Decomposition Family

Theorems 3.4 and 3.5 are interesting on their own and also widely applicable. For more precise results we need

Definition 3.6. Let $\mathcal{L}$ be a family of forbidden subgraphs, and let $p = p(\mathcal{L})$ be its subchromatic number. The **decomposition** $\mathcal{M}$ of $\mathcal{L}$ is the family of minimal graphs M for which we have $L \subseteq M \otimes K_{p-1}(r, \dots, r)$ for $r = v(L)$.[11]

In other words, $\mathcal{M}$ is the family of minimal graphs M with the property that, for some $L \in \mathcal{L}$, L contains M as an induced subgraph and $L - M$ is $(p-1)$-colorable.[12] Another way of describing $\mathcal{M}$ is that the graphs in $\mathcal{M}$ are those graphs which cannot be put into the (first) class of $T_{n,p}$ without getting forbidden subgraphs.

Let us see some examples. In case of $\mathcal{L} = \{K_{p+1}\}$ the family $\mathcal{M}$ consists of one graph, K_2. If $\mathcal{L}$ is finite, then $\mathcal{M}$ is also finite but the converse is not true: for the family of all odd cycles, $\mathcal{M} := \{K_2\}$, again.

If $L = K_{p+1}(t_0, t_1, t_2, \dots, t_p)$, $(t_0 \le t_1 \le t_2 \le \dots \le t_p)$ then $K(t_0, t_1)$ is in the decomposition and this is crucial in handling the corresponding extremal graph problem (Erdős and Simonovits [127]). If L is the icosahedron graph, then $\mathcal{M} := \{P_6, K_3 + K_3\}$, where $K_3 + K_3$ is the vertex-disjoint union of two triangles. (This is not evident!)

Generally, take any $p+1$-chromatic $L_0 \in \mathcal{L}$ and color it in $p+1$ colors. All the graphs spanned in L_0 by two colors are in $\mathcal{M}$. Hence $p(\mathcal{M}) = 1$.

The following result is due to Simonovits [246], (see also Erdős [91]).

Theorem 3.7 (Decomposition Theorem [246]). *Let $\mathcal{L}$ be a family of forbidden graphs with $p(\mathcal{L}) = p$ and decomposition $\mathcal{M}$. Then every extremal graph $S_n \in \mathbf{EX}(n, \mathcal{L})$ can be obtained from a suitable $K_p(n_1, \dots, n_p)$ by changing $O(\mathbf{ex}(n, \mathcal{M})) + O(n)$ edges. Furthermore, $n_j = \frac{n}{p} + O\left(\frac{\mathbf{ex}(n,\mathcal{M})}{n}\right) + O(1)$, and if $\mathcal{L}$ is finite, then*

$$d_{\min}(S_n) = \left(1 - \frac{1}{p}\right) n + O\left(\frac{\mathbf{ex}(n, \mathcal{M})}{n}\right) + O(1).$$

This implies that, with $m = \lceil n/p \rceil$,

$$\mathbf{ex}(n, \mathcal{L}) = e(T_{n,p}) + O\big(\mathbf{ex}(m, \mathcal{M}) + n\big).$$

If $\mathbf{ex}(n, \mathcal{M}) > cn$, then $O\big(\mathbf{ex}(m, \mathcal{M})\big)$ is sharp: put edges into the first class of a $T_{n,p}$ so that they form a $G_m \in \mathbf{EX}(m, \mathcal{M})$; the resulting graph contains no $L \in \mathcal{L}$, and has $e(T_{n,p}) + \mathbf{ex}(m, \mathcal{M})$ edges.

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[11] Taking the minimal graphs is a technical step but it guarantees, e.g., that $\mathcal{M}$ if finite when $\mathcal{L}$ is finite.

[12] $L - M$ is the graph obtained from L by deleting all the vertices of M and all the incident edges.

A second, more precise stability theorem can be established using the methods of [246], see [1]. We conclude this section with the theorem characterizing those cases where $T_{n,p}$ is the extremal graph.

Theorem 3.8 (Simonovits [247]). *The following statements are equivalent:*

(a) *The minimum chromatic number in $\mathcal{L}$ is $p+1$ but there exists (at least one) $L \in \mathcal{L}$ with an edge e such that $\chi(L-e) = p$. (Color critical edge.)*

(b) *There exists an n_0 such that for $n > n_0(\mathcal{L})$, $T_{n,p}$ is extremal.*

(c) *There exists an n_0 such that for $n > n_0(\mathcal{L})$, $T_{n,p}$ is the* **only** *extremal graph.*

The case $p = 2$ was (in some sense) settled earlier, by Erdős, see [86].

Remark 3.9. Turán's theorem is self-strengthening in the sense that one can prove easily and in an elementary way that if $T_{n,p}$ is extremal for $n > n_0$, then it is the only extremal graph for $n > n_0 + 3p$.

4. Degenerate Extremal Problems

One of the problems Turán asked in connection with his graph theorem was to find the extremal numbers for the graphs of the regular (Platonic) polytopes, see [86]. The Tetrahedron graph is K_4: the answer is given by Turán Theorem. The question of the Octahedron graph is solved by Theorem 5.2, the problems of the Icosahedron [249] and Dodecahedron [247] can be found in Section 6.

In some sense, if we cannot solve an (ordinary) extremal problem, the reason is that either it is degenerate and too difficult, or it *reduces* to a difficult degenerate extremal problem. In this sense, the central problems in Extremal Graph Theory are the degenerate ones. We return to this question in Section 5.

A subcase of the "unsolvable" degenerate extremal graph problems is when we have a probably sharp upper bound but no hope for a construction to provide a matching lower bound. This is the case for $K(p,p)$, $p \geq 4$, or for $C_{2\ell}$, for $2\ell \neq 4, 6, 10$, and probably for the cube Q_8. (I do not know of any case of a finite $\mathcal{L}$ where the "random graph construction" provided a sharp lower bound in an ordinary extremal graph problem.) So our hope lies in finite geometric or other, say, algebraic constructions.

4.1. The Kővári–T. Sós–Turán theorem

Perhaps the two most important degenerate extremal graph problems (i.e., when $\mathcal{L}$ contains bipartite graphs) are $L := K_2(a,b)$ and $L = C_{2k}$. The Kővári–T. Sós–Turán theorem [205] solves the extremal graph problem of $K_2(a,b)$, at least, provides an upper bound, which in some cases proved to be sharp and is conjectured always to be sharp.[13] This theorem is a generalization of the C_4-problem, since $C_4 = K(2,2)$, and, on the other hand, is a special case of the Erdős–Stone theorem, apart from that we get estimates sharper than in the original Erdős–Stone case.

Theorem 4.1 (Kővári–T. Sós–Turán). *Let $2 \le a \le b$ be fixed integers. Then*

$$\mathbf{ex}\big(n, K(a,b)\big) \le \frac{1}{2}\sqrt[a]{b-1}\, n^{2-\frac{1}{a}} + \frac{1}{2}an.$$

Conjecture 4.2. *The exponent $2 - \frac{1}{a}$ is sharp: For $a \le b$*

$$\mathbf{ex}\big(n, K(a,b)\big) \ge c_a n^{2-\frac{1}{a}}.$$

This is known only for $a = 2$, by Erdős [80], Erdős, Rényi, V. T. Sós, [122], and independently W. G. Brown [54], who also showed the sharpness for $a = 3$. Random graph methods [134] show that $\mathbf{ex}\big(n, K(a,a)\big) > c_a n^{2-\frac{2}{a+1}}$. Füredi [155] (superseding a result of Mörs [229]) improved the constant in the upper bound, showing that

$$\mathbf{ex}\big(n, K(2,b+1)\big) = \frac{1}{2}\sqrt{b}n^{3/2} + O(n^{4/3}),$$

and that the constant provided by Brown's construction is sharp. While one conjectures that $\mathbf{ex}\big(n, K(4,4)\big)/n^{7/4}$ converges to a positive limit, we know only, by the Brown construction, that $\mathbf{ex}\big(n, K(4,4)\big) > \mathbf{ex}\big(n, K(3,3)\big) > cn^{5/3}$. It is unknown if

$$\frac{\mathbf{ex}\big(n, K(4,4)\big)}{n^{5/3}} \to \infty.$$

[13] A footnote of [205] tells us that the authors have received a letter from Erdős in which Erdős informed them that he also had proved most of the results of [205].

Remark 4.3. The Kővári–T. Sós–Turán theorem is applicable in many cases, in number theory and geometry.

Related literature: The survey of Guy [165], Znám [291], Hyltén-Cavallius [176], Mörs [229], Füredi [155], and Section 7.

4.2. Cycles in graphs

This part will be much shorter than it deserves to be, primarily because the first part of Bondy's beautiful survey in this volume [48] covers many of the results I would mention here.

Cycles play central role in graph theory. Many results provide conditions to ensure the existence of some cycles in graphs. Among others, the theory of Hamiltonian cycles (and paths) constitute an important part of graph theory. The Handbook of Combinatorics contains a chapter by Bondy [47] giving a lot of information on ensuring cycles via various types of conditions. Also, the books of Bollobás [29], of Walther and Voss [287] and of Voss [285] contain many relevant results.

4.3. The Erdős–Gallai theorems

One of the problems posed by Turán was the extremal problem of cycles of length m. If we exclude all the odd cycles, the extremal graph will be the Turán graph $T_{n,2}$. What are the extremal graphs if we exclude the cycles of length at least m? The answer is given by the Erdős–Gallai theorem:

Theorem 4.4 Erdős and Gallai [109]). *Let* $\mathcal{L}_m = \{C_k : k \geq m\}$. *Then*

(i) $\frac{m-1}{2}n - \frac{1}{2}m^2 < \mathbf{ex}(n, \mathcal{L}_m) \leq \frac{m-1}{2}n$ *and*

(ii) *the connected graphs* G_n *whose* 2*-connected blocks are* K_{m-1}*'s are extremal (when they exist).*

The following theorem is the twin of the previous one's.

Theorem 4.5 (Erdős and Gallai [109]). $\mathbf{ex}(n, P_m) \leq \frac{m-2}{2}n$. *The union of* $\left\lfloor \frac{n}{m-1} \right\rfloor$ *vertex disjoint* K_{m-1} *(and one smaller* K_q*) shows the sharpness:* $\mathbf{ex}(n, P_m) = \frac{m-2}{2}n + O(m^2)$.

This theorem has a sharper form, proved by Faudree and Schelp [140] and an even sharoer one by Kopylov [203]. They needed the sharper form to prove some Ramsey theorems on paths.

Remark 4.6. A "loop" is a path $x_1x_2\ldots x_k$ and an extra edge x_kx_j for some $j < k$. Beside asking the problem of paths and cycles, Turán also asked for the determination of the extremal number of "loops". The problem was solved by Andrásfai [10]. Erdős also mentioned this result in [89] and one can find a lot of information on this topic in Voss' description [287, 286].

Related literature: Faudree and Schelp [140, 141], Kopylov [203], Voss [287, 286].

4.3.1. The Erdős–T. Sós conjecture. Erdős and T. Sós observed that the same estimates hold both for the path P_m and the star $K_2(1, m-1)$ and these being two extremes among the trees of m vertices, they formulated

Conjecture 4.7 (Erdős–T. Sós)**.** *For any tree* T_m,

$$\mathbf{ex}(n, T_m) = \frac{m-2}{2}n + O(1).$$

Some asymptotic versions of this conjecture were proved by Ajtai, Komlós and Szemerédi, (unpublished), also, the conjecture is proved in its sharp form for some special families of trees, like caterpillars.

Ajtai, Komlós, Simonovits and Szemerédi proved [3]:

Theorem 4.8. *For every* $\varepsilon > 0$ *there exists an* m_0 *such that for any* $m > m_0$, *for any tree* T_m,

$$\mathbf{ex}(n, T_m) \le \frac{m-2}{2}n + \varepsilon n.$$

The Loebl conjecture, originating from "problems on discrepancy of trees", is very strongly related to these topics.

Conjecture 4.9 (Loebl)**.** *If* G_n *has at least* $\frac{1}{2}n$ *vertices of degree at least* $\frac{1}{2}n$, *then* G_n *contains all trees of at most* $\frac{1}{2}n$ *edges.*

An asymptotic version of this was proved by Ajtai, Komlós, and Szemerédi [4]. The conjecture was generalized by Komlós and T. Sós:

Conjecture 4.10. *If G_n has at least $\frac{1}{2}n$ vertices of degree at least k, then G_n contains all trees of at most k edges.*

A related theorem for paths was proved earlier by Erdős, Faudree, Schelp and me [104]. As we wrote in [104],

> We wish to determine the minimum value of ℓ, such that each graph on n vertices with at least ℓ vertices of degree at least m contains a P_{k+1}.

Conjecture 4.11 ([104]). *Let n, m and k be fixed positive integers, with $n > m \geq k$ and set $\delta = 2$ when k is even and $\delta = 1$ when k is odd. If G_n is a graph on n vertices and at least*

$$\ell := \left\lfloor \frac{k-1}{2} \right\rfloor \left\lfloor \frac{n}{m+1} \right\rfloor + \delta \tag{3}$$

vertices of degree $\geq m$, then G_n contains a P_{k+1}.

This is sharp, if true. We proved (among others) that this is essentially true:

Theorem 4.12. *For every positive integer k, Conjecture* 4.11 *holds if δ is replaced by $O(1)$ in* (3).

Problem 4.13. *Assuming the connectivity of G_n is it enough to assume roughly half of the vertices required in* (3) *to be of high degree to ensure a $P_{k+1} \subseteq G_n$?*

Related literature: Erdős, Füredi, Loebl, T. Sós [108], Ajtai, Komlós, and Szemerédi [4], Erdős, Faudree, Schelp and Simonovits [104].

4.4. The case of excluded C_{2k}

Since the odd cycles are 3-chromatic color-critical graphs, one can apply Theorem 3.8 to them to get the following Erdős result [86]: $\mathbf{ex}(n, C_{2k+1}) = \left[\frac{n^2}{4}\right]$ if $n > n_0(k)$.

The case of even cycles is much more fascinating. The upper bound would become trivial if we assumed that G_n is (almost) regular and contains no cycles of length $\leq 2k$. The difficulty comes from that we exclude *only* C_{2k}. The first case is that of C_4, see [80, 54, 122]... We shall discuss this very important problem later. The general case is described by

Theorem 4.14 (Erdős, Bondy–Simonovits [49]).

$$\mathbf{ex}(n, C_{2k}) = O(kn^{1+1/k}).$$

Theorem 4.15 (Bondy–Simonovits [49]). *If* $e(G_n) > 100kn^{1+1/k}$, *then*

$$C_{2\ell} \subseteq G_n \quad \text{for every integer} \quad \ell \in [k, kn^{1/k}].$$

Erdős stated Theorem 4.14 in [86] without proof[14] and conjectured Theorem 4.15, which we proved. The upper bound on the cycle-length is sharp: take a G_n which is the union of complete graphs of size $ckn^{1+1/k}$.

Related literature: Bondy [48], Simonovits [255] Gyárfás, Komlós, and Szemerédi [168].

4.5. Reduction to simpler graphs

As Erdős pointed out [90], the following holds.

Theorem 4.16. *Every graph* G_n *has a bipartite subgraph* $H(U, V)$ *in which each vertex has at least half of its original degree:* $d_H(x) \geq \frac{1}{2} d_G(x)$. *Thus* $e\big(H(U,V)\big) \geq \frac{1}{2} e(G_n)$.

There are several results on how large p-partite graph can one find in a G_n, but here I skip the topic. One important consequence of this (almost trivial) fact is that (as to the order of magnitude), it does not matter if we optimize $e(G_n)$ over *all* graphs or only over the *bipartite* graphs, if we care only for the exponent:

If $\mathbf{ex}_B(n, \mathcal{L})$ denotes the maximum number of edges a *bipartite* G_n can have without containing subgraphs from $\mathcal{L}$, then $\mathbf{ex}(n, \mathcal{L}) \leq 2\mathbf{ex}_B(n, \mathcal{L})$.[15] So we may always reduce our problems to bipartite G_n. To prove the cube-theorem, (see Section 4.6.1) Erdős and I developed in [126] a much less trivial reduction. We need

[14] The dependence of the constant on k was our result but it is unknown if it is sharp or not?

[15] Here we are interested only in the case $p(\mathcal{L}) = 1$, but the assertion itself holds for every $\mathcal{L}$.

Definition 4.17. G_n is d-pseudo-regular, if its maximum degree divided by its minimum degree is bounded by d.

Let $\mathbf{ex}_{d,B}(n, \mathcal{L})$ denote the maximum number of edges a d-pseudo-regular, bipartite G_n can have without containing subgraphs from $\mathcal{L}$.

Theorem 4.18 (Erdős–Simonovits [126]). *For any $\mathcal{L}$, if d is sufficiently large, and $\mathbf{ex}_{d,B}(n, \mathcal{L}) = O(n^{1+\alpha})$, then also $\mathbf{ex}(n, \mathcal{L}) = O(n^{1+\alpha})$.*

This immediately follows from

Theorem 4.19 ([126]). *Let $d = 10 \cdot 2^{\alpha^{-2}-1}$. If $e(G_n) \geq n^{1+\alpha}$, then G_n contains a d-pseudo-regular H_m with $e(H_m) \geq m^{1+\alpha}$ and $m > n^{\frac{1-\alpha}{1+\alpha}}$.*

4.6. Recursion theorems

Recursion theorems could be defined for ordinary graphs and hypergraphs, for ordinary degenerate extremal problems and non-degenerate extremal graph problems, for supersaturated graph problems, (see Erdős–Simonovits [130]) and many other similar cases. However, here we shall restrict our considerations to ordinary degenerate extremal graph problems. In this case we have a bipartite L and a procedure assigning an L' to L. Then we wish to deduce upper bounds on $\mathbf{ex}(n, L')$, using upper bounds on $\mathbf{ex}(n, L)$.

4.6.1. The "cube"-recursion. On the cube-graph we have

Theorem 4.20 (Cube Theorem, Erdős–Simonovits [126]).

$$\mathbf{ex}(n, Q_8) = O(n^{8/5}).$$

We conjectured that the exponent 8/5 is sharp. Unfortunately, no "reasonable" lower bound is known. The Cube theorem follows from a recursion theorem:

Theorem 4.21 (Recursion Theorem [126]). *Let L be a bipartite graph, colored in BLUE and RED. A $K(t, t)$ be also colored in BLUE and RED. Let L^* be the graph obtained from these two (vertex-disjoint) graphs by joining each vertex of L to all the vertices of $K(t, t)$ of the other color. If $\mathbf{ex}(n, L) = O(n^{2-\alpha})$ and*

$$\frac{1}{\beta} - \frac{1}{\alpha} = t,$$

then $\mathbf{ex}(n, L^*) = O(n^{2-\beta})$.

Applying this recursion theorem with $t = 1$ and $L = C_6$ we obtain the Cube-theorem. This Erdős–Simonovits recursion immediately provided a unified, new proof for several old theorems, and disproved an earlier conjecture of Erdős that the exponents in the degenerate extremal problems always have either the form $1 + \frac{1}{\ell}$ or $2 - \frac{1}{\ell}$, for some integer $\ell > 1$. (The counterexample was more complicated than the cube; despite that we conjecture that $\mathbf{ex}(n, Q_8) \geq c_Q n^{8/5}$, we do not know if $\mathbf{ex}(n, Q_8)/n^{3/2} \to \infty$ when $n \to \infty$.)

This recursion theorem has several interesting applications. One immediately can deduce from it the Kővári–T. Sós–Turán theorem (apart from the value of the multiplicative constant), by applying Theorem 4.21 to $L = K(1, q-p+1)$ with $t = p-1$. (Of course, in some sense this is "cheating", since we use here the extension of the technique we learned from the proof of Theorem 4.1.) However, if we apply this, e.g., to $t = p-1$ and any tree, then we get, among others

Theorem 4.22 (Erdős). *Let L be obtained from $K_2(a+1, a+1)$ by deleting an edge. Then* $\mathbf{ex}(n, L) = O\left(n^{2-\frac{1}{a}}\right)$.

4.6.2. The Θ-recursion. Faudree and I sharpened Theorem 4.14 in another direction:

Definition 4.23 (Theta-graph). $\Theta(k, p)$ is the graph consisting of p independent paths of length k (i.e. k edges) joining two vertices x and y.

Clearly, $\Theta(k, 2) = C_{2k}$. We proved

Theorem 4.24 (Faudree–Simonovits [142]). $\mathbf{ex}\big(n, \Theta(k, p)\big) < c_{k,p} \cdot n^{1+1/k}$.

Fan Chung [69] constructed graphs, *almost* showing that for $p > k_0$,

$$\mathbf{ex}\big(n, \Theta(k, p)\big) > \tilde{c}_{k,p} \cdot n^{1+1/k}.$$

The Erdős–Rényi Theorem [120] shows that Theorem 4.24 is sharp in the sense that

$$\mathbf{ex}\big(n, \Theta(k, p)\big) > c^*_{k,p} n^{1+\frac{1}{k}+\frac{1}{kp}}.$$

Again, see Bondy [48], for some further information. Theorem 4.24 was deduced from a recursion theorem. We need two definitions.

Definition 4.25 Given a bipartite graph L and a fixed proper 2-coloring of it, $\psi : L \to \{\text{Red}, \text{Blue}\}$, $L_k(L, \psi)$ is the graph obtained by taking a new vertex x outside of L and joining it to each Red vertex of L by a path of length $k-1$,[16] where these paths have no common vertices but x.

Most of our graphs are L connected and therefore they have only two 2-colorings. (The matching is not connected but it has only 1 2-coloring and is uninteresting in our theorems.)

Definition 4.26. Given a bipartite L and a fixed 2-coloring of L, $\mathbf{ex}^*(a, b, L)$ is the maximum number of edges a bipartite graph $G(a, b)$ can have without containing an L whose first color class is in the first color class of $G(a, b)$ and the second color class is in the second color class of $G(a, b)$.[17] $\mathbf{ex}^*(n, L) = \max_a \mathbf{ex}^*(a, n-a, L)$.

Theorem 4.27. *Let L be an arbitrary bipartite graph with a fixed coloring ψ and assume that*

$$\mathbf{ex}^*(n, L) = O(n^{2-\alpha}). \tag{4}$$

Then for

$$\beta = \frac{\alpha + \alpha^2 + \ldots + \alpha^{k-2}}{1 + \alpha + \alpha^2 + \ldots + \alpha^{k-2}}$$

we have

$$\mathbf{ex}\big(n, L_k(L, \psi)\big) \le \mathbf{ex}^*\big(n, L_k(L, \psi)\big) = O(n^{2-\beta}). \tag{5}$$

A version of this theorem immediately implies the following generalization of Theorem 4.24.

Theorem 4.28. *If T is a tree colored in Red and Blue, then for $L = L_k(T, \psi)$, $\mathbf{ex}(n, L) = O\left(n^{1+\frac{1}{k}}\right)$.*

Related literature: Faudree and Simonovits [142, 143], Simonovits [254], [255] and Bondy [48].

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[16] i.e., $k-1$ edges!

[17] I.e., we do not exclude the occurrence of subgraphs of the opposite position!

4.7. Conjectures on degenerate extremal problems

(a) The first two conjectures below are from Erdős and myself.

Conjecture 4.29. *Is it true that for any bipartite graph L, there exist an $\alpha \in [0,1)$ and a $c_L > 0$ such that*

$$\frac{\mathbf{ex}(n,L)}{n^{1+\alpha}} \to c_L > 0 \quad \textit{as} \quad n \to \infty?$$

Perhaps α is always rational?

The motivation is that perhaps there are always some nearly optimal graphs coming from finite geometric constructions, or analogous algebraic constructions, where the whole graph has $\approx p^m$ vertices and the neighborhoods of the vertices are some k-dimensional surfaces (perhaps slightly modified), therefore $\alpha = k/m$.

Conjecture 4.30 ([131]). *If $e(G_n) = \mathbf{ex}(n, C_4) + 1$, then G_n contains at least $\sqrt{n} + o(\sqrt{n})$ copies of C_4.*[18]

The motivation is that in the Brown–Erdős–Rényi–T. Sós graph [54, 122] this can easily be verified.

Related literature: Erdős and Simonovits [131].

(b) We close this part with a beautiful but probably difficult problem of Erdős.

Conjecture 4.31. $\mathbf{ex}(n, \{C_3, C_4\}) = \frac{1}{2\sqrt{2}} n^{3/2} + o(n^{3/2})$.

The meaning of this conjecture is that excluding C_3 beside C_4 has the same effect as if we excluded all the odd cycles. If we replace C_3 by C_5, then this is true, see [129]. Erdős risks the even sharper conjecture that the exact equality may hold:

$$\mathbf{ex}(n, \{C_3, C_4\}) = \mathbf{ex}(n, \{C_4, C_3, C_5, C_7, C_9, C_{11} \dots\}).$$

Related further literature: Survey paper of R. Guy [165] and also of Guy and Znám [166] on $K(a,b)$ and Lazebnik, Ustimenko and Woldar [207], [208].

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[18] This is a supersaturated graph problem, see Section 6.5.

4.8. A strange excluded graph

In the late 60's, early 70's we knew already many results on the C_4-free graphs. The connection between C_4-free graphs and finite geometries was also clear. Two important excluded graph sequences, the $K_2(a,b)$ and the C_{2k}, were fairly well described. The "cube" theorem was proved. Erdős wanted to understand what happens if the excluded L is C_4-free. So he asked two related questions, discussed below.

Problem 4.32. *Let B_{10} be the bipartite graph on x_1, x_2, x_3, x_4 and $y_{1,2}$, $y_{1,3}$, $y_{1,4}$, $y_{2,3}$, $y_{2,4}$, $y_{3,4}$, where $y_{i,j}$ is joined to x_i and x_j.*

(a) *Is $\mathbf{ex}(n,B_{10}) = O\big(n^{3/2-c}\big)$, for some constant $c>0$?*

(b) *If L_{11} is obtained from B_{10} by adding a new vertex z and joining it to all the x_i's, is $\mathbf{ex}(n,L_{11}) = O(n^{3/2})$?*

Clearly, $C_4 \not\subseteq B_{10}$, while L_{11} contains six C_4's. The second problem was solved by Füredi, the first one by Faudree and me.

Theorem 4.33 (Füredi [150]). $c_1 n^{3/2} \le \mathbf{ex}(n,L_{11}) \le c_2 n^{3/2}$.

The lower bound immediately follows from that $C_4 \subseteq L_{11}$. More generally, let $F(k,t)$ be the bipartite graph with k vertices $x_1,\dots,x_k$ and $\binom{k}{2}t$ further vertices in groups U_{ij} of size t, where all the vertices of $\cup U_{ij}$ are independent and the t vertices of U_{ij} are joined to x_i and x_j ($1 \le i < j \le k$). Finally, a vertex z be joined to all the x_ℓ's. Erdős asked for the determination of $\mathbf{ex}\big(n,F(k,t)\big)$ for $t=1$. For $t=1$ and $k=2$ this is just C_4, so the extremal number is $\approx \frac{1}{2}n^{3/2}$. Erdős also proved [86] for $F(3,1)$ which is the cube graph minus an edge, that $\mathbf{ex}\big(n,F(3,1)\big) = O(n^{3/2})$. (This follows from our Theorem 4.21 as well.)

Theorem 4.34 (Füredi [150]). *For every $k \ge 2$ and $t \ge 1$,*

$$\mathbf{ex}\big(n,F(k,t)\big) = O(n^{3/2}).$$

Theorem 4.35 (Faudree–Simonovits [142]). $\mathbf{ex}(n,B_{10}) = O(n^{3/2-c})$, *for some constant $c>0$.*

5. The Product Conjecture

When I started working in extremal graph theory, I formulated (and later slightly modified) a conjecture on the structure of extremal graphs in non-degenerate cases.

Conjecture 5.1 (Product structure). *Let $\mathcal{L}$ be a family of forbidden graphs and $\mathcal{M}$ be the decomposition family of $\mathcal{L}$. If no trees (or forests) occur in $\mathcal{M}$, then all the extremal graphs S_n for $\mathcal{L}$ have the following structure: $V(S_n)$ can be partitioned into $p = p(\mathcal{L})$ subsets $V_1, \dots, V_p$, with $|V_i| = \frac{n}{p} + o(n)$, so that V_i is* **completely joined** *to V_j for every $1 \le i < j \le p$.*

This implies that each S_n is the product of p graphs G_i, where each G_i is extremal for some degenerate family $\mathcal{L}_{i,n}$. The meaning of this conjecture is that (almost) all the non-degenerate extremal graph problems can be reduced to degenerate extremal graph problems, see Section 5.1. A non-trivial illustration is the Octahedron theorem:

Theorem 5.2 (Erdős–Simonovits [127]). *Let $O_6 = K_3(2,2,2)$ (i.e. O_6 is the graph defined by the vertices and edges of the octahedron.) If S_n is an extremal graph for O_6 for $n > n_0(O_6)$, then S_n is a product: $S_n = H_m \otimes H_{n-m}$ for some appropriate H_m and H_{n-m} (depending on S_n) where $m = \frac{1}{2}n + o(n)$. Further, H_m is an extremal graph for C_4 and H_{n-m} is extremal for P_3.*

The last sentence of this theorem is an easy consequence of that S_n is the product of two other graphs of approximately the same size. More generally, analogous product results hold for all the forbidden graphs $L = K_{p+1}(t_0, t_2, t_1, \dots, t_p)$ if $t_0 = 2$ or $t_0 = 3$, see [127]. Probably the octahedron theorem can be extended to all graphs $L = K_{p+1}(t_0, t_1, \dots, t_p)$ and even to more general cases. On the other hand, in [252] counterexamples are constructed to the product-conjecture if we allow a long path in the decomposition family. If the decomposition contains trees, both cases can occur: the extremal graphs may be non-products and also they may be products. Turán's theorem itself is a product-case, where the decomposition family contains $K_2 = P_2$. In all the "natural cases" the extremal graphs are products. A theorem very strongly connected to Theorem 5.2 is that of Griggs, Simonovits and Thomas [164][19]:

Theorem 5.3. *Let $\mathcal{L}_{k,\ell}$ be the family of all graphs on k points and ℓ edges. If S_n is an extremal graph for $\mathcal{L}_{6,12}$ for $n > n_0(\mathcal{L}_{6,12})$, then $S_n = H_m \otimes H_{n-m}$ for some $m = \frac{1}{2}n + o(n)$. Further, H_m is an extremal graph for $\{C_3, C_4\}$ and $e(H_{n-m}) = 0$.*

5.1. The importance of the Product Conjecture

The product conjecture helps to reduce the non-degenerate extremal problems to degenerate ones. Indeed, if S_n is extremal for $\mathcal{L}$ and $S_n = U_m \otimes W_{n-m}$, then U_m is extremal for the family $\mathcal{M}^*$ defined as the set of those graphs M for which $M \otimes W_{n-m}$ contains some $L \in \mathcal{L}$. Further, if $p(\mathcal{L}) > 1$ then $p(\mathcal{M}^*) < p(\mathcal{L})$. Generally we see that the graphs $G[U_i]$ are extremal for some $\mathcal{M}^{\#}_{i,n}$ containing $\mathcal{M}$. As we have mentioned, $p(\mathcal{M}) = 1$. Hence $p(\mathcal{M}^{\#}_{i,n}) = 1$ as well.

Two questions should be clarified here: (a) is it enough to know the decomposition family to solve an extremal graph problem exactly? (b) What is really missing to generalize the Octahedron Theorem to all $K_{p+1}(t_0, \dots, t_p)$?

(a) There are cases when the decomposition family is exactly the same for $\mathcal{L}_1$ and $\mathcal{L}_2$ but $\mathbf{ex}(n, \mathcal{L}_1) \neq \mathbf{ex}(n, \mathcal{L}_2)$. So the decomposition does not completely determine the extremal number. If this is so then we have to use some further information in our proofs as well. For the octahedron, P_3 is not in $\mathcal{M}$ but to determine the extremal graphs, Erdős and I had to use the fact that if we put a P_3 into both classes of $T_{n,2}$, then the resulting graph will contain O_6.

(b) Let us call $x \in v(L)$ a *weak point* if[20] $\mathbf{ex}^*(n, L - x) = o\big(\mathbf{ex}(n, L)\big)$. In our proof of the Octahedron theorem we used that "$K(a, b)$ has a *weak point* for $p \leq 3$" and we think this always holds but we do not know, since Conjecture 4.2 is not proved.[21]

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[19] The results of [164] are strongly connected to Erdős's paper [86], see also Sections 6.2, 6.4.

[20] See Definition 4.26.

[21] By the lower bounds in [192, 8], now we know the corresponding "product results" for every $t_1 > (t_0 - 1)!$.

Related literature: Erdős and Simonovits [127], Simonovits [248], [252], [256].

6. Some Non-degenerate Problems

6.1. The Dodecahedron and Icosahedron Problems

Let $H_{n,p,s} = K_{s-1} \otimes T_{n-s+1,p}$. The situation for the dodecahedron and icosahedron graphs is as follows:

Theorem 6.1 (Simonovits [247]). *For n sufficiently large, $H_{n,2,6}$ is the only extremal graph for the dodecahedron-graph.*

Theorem 6.2 (Simonovits [249]). *For the icosahedron-graph I_{12}, $H_{n,3,3}$ is the only extremal graph, for $n > n_0$.*

The case of the Dodecahedron graph is much simpler. Its solution led to a general theory, described in [247], involving "chromatic perturbations" of the extremal graphs. The solution of the icosahedron problem led to another theory: to the solution of problems where the decomposition $\mathcal{M}(\mathcal{L})$ contains a path. The theorem thus obtained is one of the most general results in that part of Extremal Graph Theory where $\mathbf{ex}(n,\mathcal{L}) = e(T_{n,p}) + O(n)$. It immediately solves many other problems, see e.g. [256]. The methods used to prove these two theorems also imply

Theorem 6.3 (Simonovits [256]). *For $n > n_0$, $H_{n,2,3}$ is the (only) extremal graph for the Petersen graph.*

Remark 6.4. The decomposition of the Petersen graph contains 3 independent edges.

6.2. Extremal graph problems with linear decomposition

In [247] I have given a fairly general theorem which provides an "almost algorithmic" solution of many involved extremal graph problems. This theorem (Theorem 6.8) covered most of the known results of those days with linear error terms, i.e., when $\mathbf{ex}(n,\mathcal{L}) - e(T_{n,p}) = O(n)$. It covered the case of the Dodecahedron graph and would have covered the case of the Petersen graph as well, and several generalizations of the Petersen graph, e.g., the Kneser graph, and others, except that those days I have not formulated these problems (see [256]). (In some sense, it was motivated by and covered the case of the Icosahedron as well.) Informally, the meaning of this theorem was that if the decomposition contains a path P_τ, then some extremal graphs are obtained from a $T_{n,p}$ by adding small symmetrical blocks to each class and a few new, high degree vertices, joined to these blocks very symmetrically. This theorem enables us to solve many extremal graph problems "almost algorithmically" even in the case of "chromatic perturbation". We need some definitions.

Definition 6.5. The subgraphs $H_1, \dots, H_\ell \subseteq G$ will be called *symmetric* in G if (a) there are no edges between them and (b) one can fix some automorphisms $\psi_i : H_1 \to H_i$ such that for any $y \in G - H_i - H_1$, $x \in V(H_1)$ is joined to y iff $\psi_i(x) \in V(H_i)$ is joined to y.

Definition 6.6 (Family of symmetric graphs)**.** $\mathcal{D}(n,p,r)$ is the class of graphs G_n satisfying the following symmetry condition:

(i) It is possible to omit $\le r$ vertices of G_n so that the remaining graph G^* is a product (of graphs of almost equal order):

$$G^* = \prod_{\ell \le p} G_{m_\ell} \quad \text{where} \quad \left| m_\ell - \frac{n}{p} \right| \le r.$$

(ii) For every $\ell \le p$, there exist connected graphs $H_{\ell,j} \subseteq G_{m_\ell}$ such that $H_{\ell,j}$ $\quad (j = 1, \dots, k_\ell)$ are symmetric subgraphs of G_n (with appropriate isomorphisms $\psi_{\ell,j} : H_{\ell,1} \to H_{\ell,j}$), further, $v(H_{\ell,j}) \le r$ and $G_{m_\ell} = \sum_{j \le k_\ell} H_{\ell,j}$, where the sum $\sum H_{\ell,j}$ is the vertex-disjoint union.

Before formulating our main theorem, let us generalize the ordinary extremal problems to extremal problems with chromatic perturbations. A "chromatic condition" is defined in some more general way[22]; here we shall restrict ourselves to a simpler, less general version:

Definition 6.7. Let q, Ω be two given non-negative integers. The chromatic condition $\mathcal{A} := \mathcal{A}_{q,\Omega}$ is the family of graphs from which one cannot delete (at most) Ω vertices to get a q-colorable graph.

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[22] The theorem is the same for the general case just the definition is more general.

Given a chromatic condition $\mathcal{A}$, we shall denote by $(\mathcal{L}, \mathcal{A})$ the extremal problem of determining the maximum of $e(G_n)$, assuming that G_n contains no $L \in \mathcal{L}$ and $G_n \in \mathcal{A}$.

Theorem 6.8 (Symmetrical extremal graphs, [247]). *Assume that a finite family $\mathcal{L}$ of forbidden graphs with $p = p(\mathcal{L})$, and a chromatic condition $\mathcal{A}$ are given. If for some $L \in \mathcal{L}$ and $v := v(L)$,*

$$L \subseteq P_v \otimes K_{p-1}(v, v, \dots, v), \tag{6}$$

then there exists a constant $r = r(L)$ such that, for every n, $\mathcal{D}(n, p, r)$ contains an extremal graph for $(\mathcal{L}, \mathcal{A})$.

Related literature: Simonovits [247, 256].

6.3. Large excluded subgraphs

Bollobás, Erdős, Simonovits and Szemerédi [39] considered problems where the excluded graphs were parametrized and the parameters tended to infinity as $n \to \infty$. (The Erdős–Stone theorem is also of such type.) We formulate here two theorems from that paper.

Theorem 6.9. *Let I_h be the graph consisting of h independent vertices. Let M be a bipartite graph. Put*

$$q(n, M) = \max_{n_1+n_2=n} \left\{ n_1 n_2 + \mathbf{ex}(n_1, M) + \mathbf{ex}(n_2, M) \right\}.$$

There exists a $c > 0$ and an n_0 such that if $n > n_0$ and $e(G_n) > q(n, M)$, then G_n contains an $L := M \otimes I_{[cm]}$, and every extremal graph $U_n \in \mathbf{EX}(n, L)$ is a product of an $S_m \in \mathbf{EX}(m, M)$ and an $S_{n-m} \in \mathbf{EX}(n - m, M)$, where $m = n/2 + o(n)$.

Clearly, this theorem is strongly connected (a) to the general theory and (b) to the Octahedron theorem. Another result of [39] is about the fact that if a graph does not contain short odd cycles, then it can be turned "easily" into a bipartite graph. To formulate this result more generally, let $L[t]$ denote the graph obtained by replacing each vertex of L by an independent t-tuple and joining two new vertices if the originals have been joined.

Theorem 6.10. *Fix an integer t. For every $\varepsilon > 0$ there is a $K(\varepsilon)$ such that if a graph G_n cannot be turned into bipartite by deleting at most εn^2 edges, then $C_{2\ell+1}[t] \subseteq G_n$ for some $0 < \ell \le K(\varepsilon)$.*

Our estimate for $K(\varepsilon)$ was fairly trivial, much later Komlós has found the right order of magnitude of $K(\varepsilon)$ [193].

Remark 6.11. *We gave two proofs of Theorem 6.10. One of them used the Regularity Lemma, the other was "quite elementary". Of course, using the Regularity Lemma if one gets a $C_{2\ell+1}$ then one always gets a "blown-up" odd cycle, i.e., a $C_{2\ell+1}[t] \subseteq G_n$ as well, for any fixed t and $n > n_0(t, \varepsilon)$.*

6.4. "The complete list of theorems"

Watching Erdős working, beside of his great proving power and elegance, one striking feature was, how he posed his conjectures. Often we did not immediately understand the importance of his questions. Slightly mockingly, once one of his best friends, A. Hajnal told him: "You would like to have a Complete List of Theorems". I think there is some truth in this remark, yet one modification should be made. Erdős did not like to state his conjectures immediately in their most general forms. Instead, he picked very special cases and attacked first these ones. Mostly he picked his examples "very fortunately". Therefore, having solved these special cases he very often discovered whole new areas, and it was difficult for the surrounding to understand how could he be so "fortunate". So, the reader of Erdős and the reader of this survey should keep in mind that Erdős's method was to attack first *important special cases.*

Examples of "complete lists"

The Smolenice[23] paper. Perhaps the first survey paper of Erdős I know is the Smolenice paper [86]. In the Smolenice paper Erdős defines three functions, $f_1(n, k, \ell)$, $f_2(n, k, \ell)$, and $f_3(n, k, \ell)$.[24]

[23] The Smolenice Graph Conference was one of the early ones, in Czechoslovakia, June 1963.

[24] The +1 comes from the fact that sometimes we speak of the maximum number of the edges in the $\mathcal{L}$-free G_n, in other cases about the minimum number of edges ensuring an $L_k \subseteq G_n$.

Here $f_1(n,k,\ell) = \mathbf{ex}(n, \mathcal{L}_{k,\ell}) + 1$, where $\mathcal{L}_{k,\ell}$ is the family of k-vertex graphs with ℓ edges;

$$f_2(n,k,\ell) = \min_{L_k \in \mathcal{L}_{k,\ell}} \mathbf{ex}(n, L_k) + 1, \qquad f_3(n,k,\ell) = \max_{L_k \in \mathcal{L}_{k,\ell}} \mathbf{ex}(n, L_k) + 1.$$

Erdős proved several results on these functions, and also listed all the extremal numbers $\mathbf{ex}(n, L)$ for $v(L) \le 5$. Finally, he listed various interesting results and problems, demonstrating his desire to find the "complete list".

The case of K_{p+1}. When Turán proved his theorem, Erdős and others were fascinated by its simplicity and elegance and proved later many generalizations of it. I will again illustrate his "quest for the complete list", but will *not* try to list all his theorems connected to $\mathbf{ex}(n, K_{p+1})$.

The first one is the degree-majorization theorem:

Theorem 6.12. *If G_n contains no K_{p+1} and the degrees of G_n are $d_1 \ge d_2 \ge \ldots \ge d_n$, then there exists a p-chromatic G_n^* with degrees $d_1^* \ge d_1$, $d_2^* \ge d_2, \ldots, d_n^* \ge d_n$.*

Clearly, $2e(G_n) \le \sum d_i \le \sum d_i^* = 2e(G_n^*) \le e(T_{n,p})$, (since $T_{n,p}$ has the most edges among the p-chromatic n-vertex graphs): Theorem 6.12 implies Turán Theorem, apart from the uniqueness of the extremal graph.

The following conjecture of Erdős was proved by Bollobás and Thomason [43] and Erdős and T. Sós [136]:

Theorem 6.13. *If $e(G_n) > e(T_{n,p})$, then G_n has a vertex x of degree m for which its neighborhood $\Gamma(x)$ contains more than $e(T_{m,p-1})$ edges.*

The motivation of this theorem is that it immediately implies Turán's theorem, by induction on p.

Related further literature: Bondy [46], Bollobás [32].

Rademacher type theorems. Almost immediately after Turán's result, Rademacher proved the following nice theorem (unpublished, see [84]) :

Theorem 6.14 (Rademacher Theorem). *If $e(G_n) > \lfloor \frac{n^2}{4} \rfloor$, then G_n contains at least $\left[\frac{n}{2}\right]$ triangles.*

This is sharp: adding an edge to (the larger class of) $T_{n,2}$ we get $\left[\frac{n}{2}\right]$ K_3's. Erdős generalized this result by proving the following two basic theorems [84]:

Theorem 6.15. *There exists a positive constant $c_1 > 0$ such that if $e(G_n) > \lfloor \frac{n^2}{4} \rfloor$, then G_n contains an edge e with at least $c_1 n$ triangles on it.*

Theorem 6.16 (Generalized Rademacher Theorem). *There exists a positive constant $c_2 > 0$ such that if $0 < k < c_2 n$ and $e(G_n) > \lfloor \frac{n^2}{4} \rfloor + k$, then G_n contains at least $k\left[\frac{n}{2}\right]$ copies of K_3.*

Erdős conjectured [81] and Lovász and I proved that $c_2 = \frac{1}{2}$ [213]. For further results see Moon and Moser [228], Bollobás [27, 28], and [213], [214]. Erdős also proved the following theorem, going into the other direction.

Theorem 6.17 (Erdős [97]). *If $e(G_n) = \lfloor \frac{n^2}{4} \rfloor - \ell$ and G_n contains at least one triangle, then it contains at least $\left[\frac{n}{2}\right] - \ell - 1$ triangles.*

(Of course, we may assume that $0 \le \ell \le \left[\frac{n}{2}\right] - 3$.)

Lovász–Simonovits results. Erdős also generalized Theorem 6.16 to the case of K_{p+1}'s:

Theorem 6.18 (Erdős, 1962). *There exists a constant c_p such that if $e(G_n) > \mathbf{ex}(n, K_{p+1}) + k$ for some $k \in [1, c_p n]$, then G_n contains at least*

$$k \cdot \prod_{0 \le i \le p} \left\lfloor \frac{n+i}{p} \right\rfloor$$

copies of K_{p+1}.

The meaning of this result is the following. Let $U(n,p,k)$ be the graph obtained from $T_{n,p}$ by adding k edges to it, in its largest class. Then — assuming that the new edges do not form any triangle — each of them will belong to

$$\prod_{0 \le i \le p} \left\lfloor \frac{n+i}{p} \right\rfloor$$

K_{p+1}'s. The above theorem asserts that if k is small, then this is sharp: the minimum number of K_{p+1}'s for a fixed number of edges is obtained by $U(n,p,k)$.

This result was also generalized by the Lovász–Simonovits theorem [214]. To be quite precise, before formulating the Lovász–Simonovits theorems we have to mention an important partial result of Moon and Moser [228]. Moon and Moser considered the following problem:

Problem 6.19. *What is the maximum number of triangles $t(n)$ a graph G_n can have without containing a K_4?*

Denote by $\mathbb{N}_p = \mathbb{N}_p(G)$ the number of K_p's in G, (e.g., $\mathbb{N}_3$ is the number of triangles, $\mathbb{N}_4$ the number of K_4's, etc.). They proved, among others that

Theorem 6.20 (Moon–Moser [228]). *For $k \geq 3$,*

$$\mathbb{N}_k(G_n) \geq \frac{1}{k(k-2)}\mathbb{N}_{k-1}(G_n)\left[(k-1)^2\frac{\mathbb{N}_{k-1}(G_n)}{\mathbb{N}_{k-2}(G_n)} - n\right].$$

This provides a nice recursive lower bound on $\mathbb{N}_k/\mathbb{N}_{k-1}$ and (using $\mathbb{N}_2/\mathbb{N}_1 = \frac{2E}{n}$) we get a lower bound on $\mathbb{N}_k$, see e.g. [214].

Let $S_{n,p,E}$ be defined as follows: take a $T_{m,p-1}$ and a triangle-free graph B_{n-m} with $e(B_{n-m}) = E - m(n-m)$. So $e(T_{m,p-1} \otimes B_{n-m}) = E$. Now, $\mathbb{N}_{p+1}(T_{m,p-1} \otimes B_{n-m})$ depends only on the choice of m. Choose m to minimize $\mathbb{N}_{p+1}(T_{m,p-1} \otimes B_{n-m})$. The resulting graph is not uniquely determined but take one of them and denote it by $S_{n,p,E}$.

Conjecture 6.21 (Lovász–Simonovits). *If $e(G_n) \geq E$, then*

$$\mathbb{N}_{p+1}(G_n) \geq \mathbb{N}_{p+1}(S_{n,p,E}).$$

Lovász and I could prove this conjecture for the special case when the edge-number is near to the edge-number of some Turán graph $T_{n,h}$. It can easily be that this conjecture is "slightly imprecise" i.e. should be slightly modified for some values of E. Below we formulate one of our main results in a simplified form.

Theorem 6.22. *For every p there exists an $\varepsilon_p > 0$ such that if $E := e(S_n) = e(T_{n,p}) + k$ for some $0 < k < \varepsilon_p n^2$, then $\mathbb{N}_{p+1}(G_n) \geq \mathbb{N}_{p+1}(S_{n,p,E})$.*

The proof of this theorem is rather long, probably because the extremum is not sharp enough. We have formulated several simpler results there. The theorem below for $p = 3$ was proved by Goodman [161] and easily follows for the general case from the Moon–Moser theorem [228].

Theorem 6.23. *Let* $t := t(G_n)$ *be defined by* $e(G_n) = \left(1 - \frac{1}{t}\right)\binom{n}{2}$. *For* $t > p$

$$\mathbb{N}_p(G_n) \geq \binom{t}{p}\left(\frac{n}{t}\right)^p.$$

Here the meaning of t is clear: for a Turán graph $G_n = T_{n,p}$ it is very near to the class number p: so it can be considered as the "fractional class number". We also proved a corresponding "stability theorem".

Theorem 6.24. *Let* C *be an arbitrary constant. There exist positive constants* δ *and* C' *such that if* $0 < k < \delta n^2$ *and for the* t *defined above, if*

$$\mathbb{N}_p(G_n) \leq \binom{t}{p}\left(\frac{n}{t}\right)^p + Ckn^{p-2},$$

then there exists a $K_p(n_1, \dots, n_p)$ *such that* G_n *can be obtained from this* $K_p(n_1, \dots, n_p)$ *by adding at most* $C'k$ *edges to it and deleting at most* $C'k$ *edges from it.*

(a) One can automatically extend Theorem 6.24 to the case $-\delta n^2 < k < \delta n^2$, (b) and one can immediately see that $\left|n_i - \frac{n}{p}\right| \leq C'\sqrt{k}$. (c) The importance of this stability theorem lies (partly) in that it can replace the application of Szemerédi's Regularity Lemma in several cases: mostly, when the conjectured extremal graph is an almost Turán graph.

Related further literature: Lovász and Simonovits [214], Lovász [212].

6.5. Supersaturated graphs: the general case

Again, this section will be much shorter than what it deserves. The case of complete graphs is fairly well described by Bollobás [28, 27], and Lovász–Simonovits, [213, 214]. The question of supersaturated graphs for bipartite excluded graphs ($p(\mathcal{L}) = 1$) is quite different. It would deserve a whole chapter. A survey of Erdős and myself on this topic is in the Waterloo volume [131], and also a "continuation" of this paper is in the same volume, by me [254]. Here I formulate only one conjecture.

Conjecture 6.25 (Erdős–Simonovits [131])**.** *If* L *is bipartite,* $e := e(L)$, $v := v(L)$, $c > 0$ *and* $E := e(G_n) > (1 + c) \cdot \mathbf{ex}(n, L)$, *then* G_n *contains at least*

$$c' \frac{E^e}{n^{2e-v}}$$

copies of L, *for some* $c' > 0$.

The meaning of this conjecture is a phase transition, and it says that if one has more edges than the extremal graph then it has (up to a multiplicative constant) at least as many L as the corresponding random graph. The conjecture has several weaker and sharper versions.

There are many results on hypergraphs as well. Supersatured graph theorems can be used also to prove ordinary extremal graph results.

Related further literature: Sidorenko [245], Brown and Simonovits [59].

7. Finite Geometries and Extremal Graph Theory

For several years random graph methods and finite geometries were the only tools to show the sharpness of certain upper bounds in degenerate extremal graph problems (or related non-degenerate extremal graph problems).[25]

Clearly, Erdős was very interested in applications and characterizations of Finite Geometries. This is not the best place to write about this: I will restrict myself to the applications in Extremal Graph Theory and mention some further related papers.

For me, the application of finite geometries in Extremal Graph Theory started in the Erdős paper, with the construction of Eszter Klein [80], to prove the sharpness of Theorem 2.1. Later Reiman returned to this topic [238] and then, much later, Erdős and Rényi [121] used finite geometry for a diameter-extremal problem. That was followed by the Erdős–Rényi–T. Sós paper [122], and by the Brown paper [54].

The connection between finite planes and the C_4-extremal problem is clear: the last paper [54] is more surprising and it is even more surprising that it yields a sharp result, see Füredi [156].

There was an alternative line, which I feel is very interesting, namely, the problem of cycles, which was strongly connected to the so called "cage"-problem.

[25] To be precise, there existed a third method as well, which I regarded very near to the finite geometric method. Also, in most cases (interesting from the point of view of extremal graph theory) the Random Graph Method gave only good approximation but sufficiently good lower bounds.

Let us return to Theorem 4.14. Is it sharp? Finite Geometric (and other) constructions show that for $k = 2, 3, 5$ YES. (See Singleton [260], Benson [22], Wenger [289] ...). Unfortunately, nobody knows if this is sharp for C_8, or for other C_{2k}'s.

In Erdős's combinatorics the finite geometries pop up in many further places, e.g., in de Bruijn-Erdős theorem, which is also connected to the Gallai-Sylvester theorem.

Several problem papers of Erdős speak of finite geometries. Here I mention only Erdős [100], Erdős, Mullin, Sós, Stinson, [118], ...

Finite Geometries and Polarized Partition Relations can also be found in Chvátal [75], Berge and Simonovits [23] Sterboul [267], where the C_4-free partitions of the complete graphs are constructed. This again is strongly connected to Erdős's combinatorics but I skip the details.

For some use of finite geometries for hypergraphs see Brown–Erdős–Sós [56], Simonovits [251], ..., T. Sós [261], and in some sense, also Berge–Simonovits [23].

An interesting result on the connection between Finite Geometries and the C_4-extremal graph problem is the so called "Friendship" theorem [122].

Related literature: Erdős, [100, 101], T. Sós [261].

7.1. Algebraic approach

Several people felt that the potential use of finite geometric construction is fairly limited: perhaps some more algebraic approach (say, the use of algebraic geometry) would provide new, beautiful results. This happened when Kollár, Rónyai and Szabó proved [192][26]

Theorem 7.1. *Fix a $b > a!$. For $n > n_0(a)$, Theorem 4.1 is sharp:*

$$\mathbf{ex}\big(n, K_2(a,b)\big) > c_p n^{2-\frac{1}{a}}.$$

This was slightly improved by Alon, Rónyai and Szabó [8]: the condition $b > a!$ was replaced by $b > (a-1)!$.

[26] Perhaps one of the first applications of clearly algebraic methods to provide lower bounds in an extremal graph problem is Margulis' Construction 9.3.

Remark 7.2. Describing walks and cycles in graphs is perhaps one of those parts of extremal graph theory where algebraic methods may be applied more often than in other extremal problems. One of the reasons for this is that the number of walks and also the number of even cycles in a graph can well be described by matrices and often by eigenvalues. I warmly recommend Noga Alon's chapter: Tools from Higher Algebra [7], which provides a lot of interesting and useful information — among others — on topics I had to describe very shortly.

The reader could ask: what is the real difference between the finite geometric and the more algebraic constructions? The question is of course, partly philosophical: in both cases the vertices of the graph form some algebraic structures, and we define the edges of a graph by some equations: to prove the non-existence of a subgraph L is reduced to the proof of that some system of equations cannot be solved. Yet, in what I call finite geometric constructions, there is a geometric idea behind the system of equations, maybe even a geometric fact, while in the more algebraic construction this picture seems to disappear.

Remark 7.3. Lazebnik, Ustimenko and Woldar provided several algebraic constructions in connection with the girth problem. Since the cycles are thoroughly explained in [48], here I skip these results, just mentioning their papers, e.g., [207, 208] and also a paper of Wenger [289] and [160].

Remark 7.4. There are other algebraic methods as well. Further, in some cases it is not so easy to decide if the construction we consider is more algebraic or more number theoretic. One important construction to be mentioned is that of Biggs and Hoare [24], a cubic graph G_n of which Weiss [288] has proved that its girth is $\geq \left(\frac{4}{3}+o(1)\right)\log n$.

8. Erdős–Pósa Theorem

8.1. Undirected case

The following question of Dirac and Erdős [87] is motivated partly by Menger Theorem. We shall call $L_1,\dots,L_k \subseteq G$ *vertex-independent*, (or simply, independent) if no two of them have common vertices.

If G is a graph

(∗) not containing two vertex-independent cycles,

how many vertices are needed to represent all the cycles of G?

K_5 satisfies (∗) and we need at least 3 vertices to represent all its cycles. Bollobás [25] proved that in all the graphs satisfying (∗) there exist 3 vertices the deletion of which results in a tree (or forest). More generally,

Let $RC(k)$ denote the minimum t such that if a graph G contains no $k+1$ independent cycles, then one can delete t vertices of G ruining all the cycles of the graph. Determine $RC(k)$!

Theorem 8.1 (Erdős and Pósa [119]). *There exist two positive constants, c_1 and c_2 such that*

$$c_1 k \log k \le RC(k) \le c_2 k \log k.$$

The lower bound used *random methods*, which can be replaced by the Margulis graph [221]. The Biggs–Hoare–Weiss construction also can be used here. This theorem is strongly connected to the following extremal graph theoretical question:

Assume that G_n is a graph in which the minimum degree is D. Find an upper bound on the girth of the graph in terms of D.

Here the usual upper bound is $\approx \dfrac{2\log n}{\log(D-1)}$.

8.1.1. The directed case. (a) The most important related result is the extension of Erdős–Pósa theorem to the directed case: this was done (after several preparatory steps) by B. Reed, N. Robertson, P. Seymour and R. Thomas [237].[27] The first positive result was

Theorem 8.2 (McCuaig [223]). *If $\vec{D}$ is a digraph in which any two directed cycles have a common vertex, then the directed cycles can be represented by 3 vertices.*

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[27] Gallai has formulated this problem for the very special case when any two directed cycles have common vertices: I assume that Gallai meant the general problem as well, just wanted to restrict himself to the simplest unsolved case. Five years later Younger published the general question.

Theorem 8.3 (Reed–Robertson–Seymour–Thomas [237]). *If $\vec{D}$ is a digraph not containing $k+1$ vertex-independent directed cycles, then the directed cycles can be represented by $O_k(1)$ vertices.*

(b) There are results, due to Dirac [78], W. G. Brown [53] and Lovász [209], describing the structure of (undirected) graphs not containing two independent cycles, and actually, McCuaig completely describes the structure of *digraphs* without 2 independent dicycles.

9. "Random Constructions?"

Probabilistic method was one of the most important methods to prove the existence of complicated combinatorial objects. As I wrote, I will deal with this subject only superficially. The reader is refered to the books of Erdős and Spencer [134], of Bollobás [30], of Alon and Spencer [9] of Spencer [265], and, of course, to the original papers of Erdős and his coauthors, among others, to the "Art of Counting" [95]. A very recent book on Random Graphs is that of Janson, Łuczak and Ruciński [180].

Erdős wrote two papers with the title "Graph Theory and Probability", one in 1959 [82], and the other in 1961, [83]. These papers were of great importance. The results of these papers may seem to be purely Ramsey theoretical, fairly surprising in those days, but they have several important consequences in Extremal Graph Theory as well. One important corollary of the main theorem of [83], more precisely, of its proof is

Theorem 9.1. *If $\mathcal{L}$ contains no trees, neither forests, then $\mathbf{ex}(n,\mathcal{L}) > c^*_{\mathcal{L}} n^{1+c_{\mathcal{L}}}$, for some constants $c^*_{\mathcal{L}}, c_{\mathcal{L}} > 0$.*

On the other hand, it is easy to see that if $L \in \mathcal{L}$ is a tree (or a forest), then $\mathbf{ex}(n,\mathcal{L}) = O(n)$. Another important corollary of the above "random construction" is the lower bound in the Ramsey–Turán type Erdős–T. Sós theorem on $\mathbf{RT}\big(n, K_{2\ell+1}, o(n)\big)$.

In some other cases the following theorem, from Graph Theory and Probability II is useful:

Theorem 9.2 ([83]). *There exist a constant $c > 0$ and an n_0 such that for $n > n_0$ there exist graphs G_n with $K_3 \not\subseteq G_n$ and $\alpha(G_n) \le c\sqrt{n}\log n$.*

The Random Graph method is definitely one which is very strongly connected to Erdős's name and was extremely successful in many cases.

Here we speak of proving the existence of some objects by random methods. Though random codes were already used by Shannon and random graphs were investigated by others as well, definitely, in modern combinatorics it was the "Erdős method". Later several variants were established, some connected to Erdős and Rényi, others to the Lovász Local Lemma [116]. Also there was another line, through Ajtai, Komlós and Szemerédi, then Rödl. All these methods can be used among others to show the existence of certain objects. One of the most important such objects was the expander graph. (I should also mention Janson's and Suen's inequalities, see e.g., [178, 179], [269], or Boppana and Spencer [50], and, finally, the use of martingale methods.)

Beside its pure theoretical interest, the fast development of Theoretical Computer Science and the great difficulties of producing truly random bits put a pressure on mathematicians to try to replace the "random constructions" by deterministic ones as often as possible.

Today we could say, that many of the "Random Constructions" were replaced by *deterministic* ones. The existence of graphs with high girth and high chromatic number (Erdős [83]) was proved *without random graphs* first by Lovász [211], (where Lovász extended the problem to hypergraphs and thus succeeded in using a — rather involved — induction), then by Nešetřil and Rödl [231, 232], then by Křiž [206] and the Margulis-Lubotzky-Phillips-Sarnak construction is also a deterministic construction to settle (among others) this problem as well.

9.1. The Margulis–Lubotzky–Phillips–Sarnak constructions

Sometimes we insist on finding **constructions** for certain cases, even when the randomized methods work easily. One such case was the **girth** problem discussed above, with one exception. Namely, in the girth problem Margulis [221, 222] and Lubotzky–Phillips–Sarnak [215, 216], succeeded in constructing regular graphs G_n of (arbitrary high) but fixed degree d and girth at least $c_d \log n$. The original random-graph existence proof is due to Erdős and Sachs [124].

Theorem 9.3 (Margulis [221]). *For every $\varepsilon > 0$ we have infinitely many values of r, and for each of them an infinity of regular graphs X_j of degree $2r$ with girth*

$$g(X_j) > \left(\frac{4}{9} - \varepsilon\right) \frac{\log v(X_j)}{\log r}.$$

Margulis also explained, how the above graphs can be used in constructing certain (explicit) error-correcting codes and generalized this construction (in the same paper) to arbitrary even degrees. The next breakthrough was due to Margulis [222] and to Lubotzky, Phillips and Sarnak [215, 216]. The graph of Lubotzky, Phillips and Sarnak was obtained *not* for extremal graph purposes. The authors, investigating the extremal spectral gap[28] of d-regular graphs, constructed graphs where the difference between the first and second eigenvalues is as large as possible. Graphs with large spectral gaps are good expanders, and this was perhaps the primary interest in [216] or in [222]. As the authors of [216] remarked, Noga Alon turned their attention to the fact that their graphs can be "used" also for several other, classical purposes.

Definition 9.4. Let X be a connected k-regular graph. Denote by $\lambda(X)$ the second largest eigenvalue (in absolute value) of the adjacency matrix of X. A k-regular graph on n vertices, $X = X_{n,k}$, will be called a **Ramanujan graph**, if $\lambda(X_{n,k}) \le 2\sqrt{k-1}$.

I do not have the place here to go into details, but the basic idea is that random graphs have roughly the spectral gap required above [159] [147], and vice versa: if the graph has a large spectral gap,[29] then it may be regarded in some sense, as if it were a random graph. So the Ramanujan graphs provide near-extremum in some problems, where random graphs are near-extremal. (See also [6], [74])

Let p, q be distinct primes congruent to 1 mod 4. The Ramanujan graph $X^{p,q}$ of [215] is a $p+1$-regular Cayley graph of $PSL(2,\mathbb{Z}_q)$ if the Legendre symbol $\left(\frac{p}{q}\right) = 1$ and of $PGL(2,\mathbb{Z}_q)$ if $\left(\frac{p}{q}\right) = -1$. ($\mathbb{Z}_q$ is the field of integers mod q.) Here we restrict ourselves to the case $\left(\frac{p}{q}\right) = 1$.

Theorem 9.5 (Alon, quoted in [216])**.** *Let $X_{n,k} = X^{p,q}$ be a non-bipartite Ramanujan graph; $\left(\frac{p}{q}\right) = 1$, $k = p+1$, $n = q(q^2-1)/2$. Then the independence number $\alpha(X_{n,k}) \le \frac{2\sqrt{k-1}}{k}n$.*

Corollary 9.6 ([216])**.** *If $X_{n,k}$ is a non-bipartite Ramanujan graph, then $\chi(X_{n,k}) \ge \frac{k}{2\sqrt{k-1}}$.*

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[28] = difference between the largest and second largest (in absolute value) eigenvalues

[29] For a more precise description see [159], [74]. If we regard a d-regular graph, then we require that the second largest (by absolute value) eigenvalue be $o(d)$: for random graphs it is around $O\left(\sqrt{d}\right)$.

Margulis, Lubotzky, Phillips and Sarnak have constructed Ramanujan graphs which are $p+1$-regular, **non-bipartite Ramanujan graphs** with $n = q(q^2-1)/2$ vertices, and with

$$\mathbf{girth}(X_{n,p+1}) \ge \frac{2}{3}\frac{\log n}{\log p} + O(1), \qquad \alpha(X_{n,p+1}) \le \frac{2\sqrt{p}}{p+1}n,$$

$$\chi(X_{n,p+1}) \ge \frac{p+1}{2\sqrt{p}}.$$

Putting $p = \text{const}$ or $p \approx n^c$ we get constructions of graphs the existence of which were known earlier only via random graph methods. Surprisingly, they are better than the known "random constructions", showing that

$$\mathbf{ex}(n, C_{2k}) > c_k n^{1+\frac{4}{3k+25}}.$$

Related literature: Lubotzky [217], Sarnak [242].

10. Topological Subgraphs

Given a graph L, we may associate with it all its topologically equivalent forms. Slightly more generally, let $\mathcal{T}(L)$ be the set of graphs obtained by replacing some edges of L by "hanging chains", i.e., paths all inner vertices of which have degree 2.

Problem 10.1. *Find the maximum number of edges a graph G_n can have without containing subgraphs from $\mathcal{T}(L)$.*

Mader [219] showed that

Theorem 10.2. *There exists a constant $c > 0$ such that if $e(G) > tn$, then G contains a topological complete p-graph $L \in \mathcal{T}(K_p)$ for $p = \left[c\sqrt{\log t}\right]$.*

This immediately implies

Corollary 10.3. *For every L, $\mathbf{ex}\big(n, \mathcal{T}(L)\big) = O(n)$.*

One could ask: how large topologically complete subgraph is guaranteed by $e(G_n) > tn$. Mader [219], Erdős and Hajnal [113], conjectured that the answer is $\approx c\sqrt{t}$. Komlós and Szemerédi proved this apart from a log-power [201], and then the sharp result was proved by Bollobás and Thomason [45] and by Komlós and Szemerédi [202]:

Theorem 10.4 ([45, 202]). *There is a positive c_1 such that if $e(G_n) > tn$, then G_n contains an $L \in \mathcal{T}(K_p)$ with $p > c_1\sqrt{t}$.*

Related literature: Bollobás, Chapter 7 of [29].

11. Extremal Subgraphs

Until now we were interested in the case when we tried to maximize $e(G_n)$ for $G_n \subseteq K_n$, assuming that G_n contains no subgraphs from $\mathcal{L}$. More generally, one could ask:

Given a sequence H_n of graphs, determine

$$\mathbf{ex}(n, H_n, \mathcal{L}) := \max\left\{ e(G_n) \ : \ G_n \subseteq H_n,\ L \not\subseteq G_n \text{ if } L \in \mathcal{L} \right\}.$$

There are several important subcases of this family of problems. One of them is the "extremal substructures of the cube". Here I will skip this field for two reasons: (a) It seems to me that the phenomena are the least understood here, (b) the topic is described in details in Bondy's paper [48] in this volume.

Related literature: Chung [71], Bollobás, Erdős, and Szemerédi [41], Bollobás, Erdős and Straus [40].

11.0.1. Subgraphs of random graphs.

Babai, Spencer and I (sharpening some results of Frankl and Rödl), proved some theorems on extremal subgraphs of random graphs [16]. Here I restrict myself to the simplest cases.

The simplest case was when $L = K_3$ and $p > 0$ was fixed.

Theorem 11.1 ([16]). *There exists a $p_0 < \frac{1}{2}$ such that if $G_{n,p}$ is a random graph with edge-probability $p > p_0$ and F_n is a K_3-free subgraph of maximum number of edges, B_n is a bipartite subgraph with maximum number of edges, then $e(F_n) = e(B_n)$ almost surely. Moreover, all triangle-free subgraphs F_n with maximum number of edges are almost surely bipartite.*

The case of C_{2k+1} was covered by the Babai–Simonovits–Spencer theorem, but only for fixed positive probability p. Haxell, Kohayakawa and

Łuczak [173] extended this result to very low probabilities at the price of some sharpness of the estimates. Kohayakawa, Kreuter and Steger considered the case of degenerate extremal problems, above all, the case of C_{2k} [190], again for low probabilities.

Related literature: Erdős, Janson, Łuczak, Spencer, [114], Łuczak, [218].

12. Typical $\mathcal{L}$-free Graphs

Erdős, Kleitman and Rothschild [115] started investigating the following problem:

How many labelled $\mathcal{L}$-free graphs exist on n vertices.

Denote the family of $\mathcal{L}$-free graphs of order n by $\mathcal{P}(n, \mathcal{L})$. We have a trivial lower bound on $\left|\mathcal{P}(n, L)\right|$: take any fixed extremal graph S_n and take all the $2^{\mathbf{ex}(n,L)}$ subgraphs of it:

$$\left|\mathcal{P}(n, L)\right| \geq 2^{\mathbf{ex}(n,L)}. \tag{7}$$

Erdős conjectured that this is mostly sharp: $\left|\mathcal{P}(n, L)\right| = 2^{(1+o(1))\mathbf{ex}(n,L)}$. In most cases it is irrelevant if we count *labelled* or *unlabelled* graphs.

Theorem 12.1 (Erdős–Kleitman–Rothschild [115]). *The number of K_p-free graphs on n vertices and the number of $p-1$-chromatic graphs on n vertices are in logarithm asymptotically equal: For every $\varepsilon(n) \to 0$ there exists an $\eta(n) \to 0$ such that if $\mathcal{P}(n, K_p, \varepsilon)$ denotes the family of graphs of n vertices and with at most εn^p subgraphs K_p, then*

$$\mathbf{ex}(n, K_p) \leq \left| \log \mathcal{P}(n, K_p, \varepsilon)\right| \leq \mathbf{ex}(n, K_p) + \eta n^2.$$

In other word, we get “a large part of them” by simply taking all the $(p-1)$-chromatic graphs.

More generally, Erdős, Frankl and Rödl [107] proved that

Theorem 12.2. *If $\chi(L) > 2$, then*

$$\left|\mathcal{P}(n, L)\right| = 2^{\mathbf{ex}(n,L)+o(\mathbf{ex}(n,L))}.$$

The corresponding question for bipartite graphs is unsolved. Even for the simplest non trivial case, i.e. for C_4, the results are unsatisfactory. This is not so surprising. All these problems are connected with random graphs, where for low edge-density the problems often become much more difficult. Kleitman and Winston [189] showed that $|\mathcal{P}(n, C_4)| \le 2^{cn\sqrt{n}}$, but the best value of the constant c is unknown. The truth should be, of course (at least, by Erdős),

$$|\mathcal{P}(n, C_4)| = 2^{((1/2)+o(1))n\sqrt{n}}.$$

Kolaitis, Prömel and Rothschild [191] sharpened Theorem 12.1: they proved that, in fact, almost every K_{p+1}-free graph is p-colorable. ("G is p-colorable" means that $\chi(G) \le p$.)

Theorem 12.3. *Let $\mathcal{C}_n(p)$ be the set of labeled p-colorable graphs on $\{1, \dots, n\}$. Then*

$$\frac{|\mathcal{P}(n, K_{p+1})|}{|\mathcal{C}_n(p)|} \to 1 \quad \text{as} \quad n \to \infty.$$

Later, Prömel and Steger [233] extended Theorem 12.1. To state their result, we remind the reader that an edge e of L is *color-critical* if $\chi(L-e) < \chi(L)$, and that many results for K_{p+1} can be generalized to graphs having critical edges.

Theorem 12.4. *Let $\chi(L) = p + 1$. Then*

$$\frac{|\mathcal{P}(n, L)|}{|\mathcal{C}_n(p)|} \to 1 \quad \text{as} \quad n \to \infty$$

if and only if L contains a color-critical edge.

Many new results concerning this area were proved in the last few years. One of them is our improvement of the Erdős–Frankl–Rödl theorem. Balogh and Bollobás and I proved [19] that

Theorem 12.5. *For every nontrivial*[30] *family $\mathcal{L}$ of graphs there exists a constant $\gamma = \gamma_{\mathcal{L}} > 0$ such that*

$$|\mathcal{P}(n, \mathcal{L})| \le 2^{\frac{1}{2}\left(1-\frac{1}{p}\right)n^2 + O(n^{2-\gamma}\log n)}, \tag{8}$$

[30] i.e. where $e(L) > 0$ for every $L \in \mathcal{L}$.

for $p = \min_{L\in\mathcal{L}} \chi(L) - 1$.

Assume that $\mathcal{M}$ is the decomposition class of $\mathcal{L}$ and $\mathbf{ex}(n,\mathcal{M}) = O(n^{2-c})$. Then we may take $\gamma = c$ in the above theorem. We also have some sharper results but I skip the details here. I finish with a related problem of Erdős.

Problem 12.6 (Erdős)**.** *Determine or estimate the number of* **maximal** *triangle-free graphs on* n *vertices.*

Some explanation. In the Erdős–Kleitman–Rothschild case (for K_3) the number of bipartite graphs was large enough to give a logarithmically sharp estimate. Here $K(a, n-a)$ are the maximal *bipartite* graphs, their number is negligible. This is why the situation becomes less transparent.

The case of induced subgraphs is discussed, e.g., in Prömel–Steger [234] Bollobás and Thomason [44], or Balogh, Bollobás and Weinreich [20].

13. Pentagonlike Structures

Many problems on triangle-free graphs are connected to the pentagon-structure. We shall call a graph "pentagonlike" if we can partition its vertices into $\mathcal{C}_1,\dots,\mathcal{C}_5$ so that all the edges join some $\mathcal{C}_i$ to $\mathcal{C}_{i+1}$ where the indices are counted mod 5: $\mathcal{C}_6 := \mathcal{C}_1$. The **maximal pentagonlike structures** are obtained when $\mathcal{C}_i$ is joined to $\mathcal{C}_{i+1}$ completely. $C_5[n_1,\dots,n_5]$ denotes the (maximal) pentagonlike graph when $|\mathcal{C}_i| = n_i$. Denote by $C_5[n]$ this graph if $\left|n_i - \lfloor n/5\rfloor\right| < 1$.[31] Disregarding $T_{n,2}$, these are the simplest triangle-free structures. There are many problems where pentagonlike structures provide the extremal configurations. Denote by $D(G)$ the minimum number of edges to be deleted from G to get a bipartite graphs.

Conjecture 13.1 (Erdős)**.** *If* $K_3 \not\subseteq G_n$, *then* $D(G_n) \le n^2/25$.

The maximal pentagonlike graph $Q_n := C_5[n]$ shows that, if true, this conjecture is sharp. The conjecture is still open, in spite of the fact that good approximations of its solutions were obtained by Erdős, Faudree, Pach and Spencer [105].

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[31] Actually, this is not uniquely defined but that does not matter here.

Theorem 13.2. *For every triangle-free graph G with n vertices and m edges*

$$D(G_n) \le \max\left\{\frac{1}{2}m - \frac{2m(2m^2-n^3)}{n^2(n^2-2m)},\ m - \frac{4m^2}{n^2}\right\} \tag{9}$$

This proves the conjecture for $e(G_n) > \frac{n^2}{5}$. The general conjecture is still open for $\frac{2n^2}{25} < e(G_n) < \frac{n^2}{5}$. The next theorem of Erdős, Győri and myself [112] states that if $e(G_n) > \frac{1}{5}n^2$, then the pentagon-like graphs need the most edges to be deleted to become bipartite. (Our result provides also information on the near-extremal structure.)

Theorem 13.3. *If $K_3 \not\subseteq G_n$ and $e(G_n) \ge \frac{n^2}{5}$, then there is a pentagonlike graph H_n^* with $e(G_n) \le e(H_n^*)$, for which $D(G_n) \le D(H_n^*)$.*

13.0.2. The number of C_5's in triangle-free graphs. A beautiful conjecture of Erdős states that if $K_3 \not\subseteq G_n$, then G_n contains at most $\left(\frac{n}{5}\right)^5$ copies of C_5. The motivation is that $C_5[n]$ contains $\approx \left(\frac{n}{5}\right)^5$ copies of C_5. In an elegant, short note E. Győri proved

Theorem 13.4 (Győri [169])**.** *Let G_n be a triangle-free graph. Then the number of C_5's in G_n is at most* $1.03 \times \left(\frac{n}{5}\right)^5$.

This was improved by Z. Füredi (in a "far from trivial way") to

Theorem 13.5 (Füredi [153])**.** *Let G_n be a triangle-free graph. Then the number of C_5's in G_n is at most* $1.003 \times \left(\frac{n}{5}\right)^5$.

13.1. Perturbation problems: Minimum degree problems

Given a family $\mathcal{L}$, the Zarankiewicz problem [290][32] asks for the determination of $\mathbf{dex}\,(n,\mathcal{L}) = \max\left\{d_{\min}(G_n) : G_n \not\supseteq L \in \mathcal{L}\right\}$, that is, the maximum minimum valence of an $\mathcal{L}$-free G_n. Since $d_{\min}(G_n) \le (2/n)e(G_n)$,

$$\mathbf{dex}\,(n,\mathcal{L}) \le \frac{2}{n}\mathbf{ex}(n,\mathcal{L}) \le \left(1-\frac{1}{p}\right)n + o(n).$$

[32] Usually, we call two different questions as Zarankiewicz problem: the other is the determination of $\mathbf{ex}\big(n, K(p,q)\big)$.

On the other hand, $T_{n,p}$ shows that this is sharp. This means that in solving Turán's problem, we have also solved Zarankiewicz's problem. Theorem 6.8 implies that chromatic perturbation in Turán's theorem changes the maximum only negligibly–that is, by $O(n)$. The chromatic perturbation in Zarankiewicz-type problems is interesting because it changes the maximum by cn^2 edges:

Theorem 13.6 (Andrásfai, Erdős and T. Sós [11]).

$$\max\left\{d_{\min}(G_n)\,:\,\chi(G_n)\geq p+1,\ K_{p+1}\not\subseteq G_n\right\}$$
$$=\left(1-\frac{1}{p-(1/3)}\right)n+0(1). \qquad \blacksquare$$

The extremal graphs have fairly transparent structures: for K_3 they are pentagonlike graphs, for K_{p+1} we take (approximately) $C_5[t]\otimes T_{n-5t,p-1}$, where $t\approx\frac{n}{3p-4}$.

Erdős and Simonovits extended the above result to arbitrary L with critical edges. The surprising phenomenon is that in all other cases we have smaller upper bounds:

Theorem 13.7 (Erdős and Simonovits [128]). *Let $L\neq K_{p+1}$, $p:=\chi(L)-1$. Assume that L has a critical edge. If G_n is L-free, and $\chi(G_n)\geq p+1$, then*

$$d_{\min}(G_n)\leq\left(1-\frac{1}{p-(1/2)}\right)n+O(1),$$

and this is sharp.

The case of K_3, high chromatic number. Define $\psi(n,L,t)$ as the minimum m such that if $d_{\min}(G_n)\geq m$, and $\chi(G_n)\geq t$, then $L\subseteq G_n$. The general problem of determining $\psi(n,K_{p+1},t)$, $t>p+1$, is much more difficult. We conjectured that $\psi(n,K_3,t)\approx n/3$, for $t\geq 4$.

Let us restrict ourselves to this simplest case.

Theorem 13.8 (Häggkvist [174]). *If G_n is triangle-free and $d_{\min}(G_n)\geq\frac{3}{8}n$, then G_n is pentagonlike.*

Several conjectures were formulated and proved or disproved in this field. A. Hajnal (using the Kneser graph) [128] gave a construction showing that $\psi(n, K_3, t) \geq \left(\frac{1}{3} - o(1)\right) n$. Perhaps today one of the most informative source on the situation is the paper of Guoping Jin's [181].

It seems to me that the next theorem was the breakthrough: Jin has defined a sequence of graphs $F(d)$ as follows: the vertices are $x_1, \dots, x_{3d-1}$ and the edges are all the edges of the corresponding C_{3d-1}, i.e., $x_i x_{i+1}$, where $x_{3d} := x_1$, and also the "diagonals" $x_i x_{i+3}$, $x_i x_{i+6}$, $\dots x_i x_{i+3[d/2]-2}$. So, $F_1 = K_2$, $F_2 = C_5$. A graph is of $F(d)$-type if it can be homomorphically mapped into $F(d)$.

Theorem (Guoping Jin [181]). *If G_n is triangle-free and $d_{\min}(G_n) > \left\lfloor \frac{(d+1)n}{3d+2} \right\rfloor$, then G_n is of F_d type. Further, this is sharp for $1 \leq d \leq 9$: there is a triangle-free non-$F(d)$-type graph with $d_{\min}(G_n) = \left\lfloor \frac{(d+1)n}{3d+2} \right\rfloor$.*

For a sharper theorem see Guoping Jin [182], for a short proof of the original Andrásfai–Erdős–T. Sós theorem, see Brandt [52]. In a recent paper, C. Thomassen [275] proved the following related result:

Theorem 13.10. *Let $c > 1/3$ be any fixed real number. Then the triangle-free graphs of minimum degree $> cn$ (where n is the number of vertices) have bounded chromatic number.*

14. Anti-Ramsey Theorems

Perhaps the first person I heard speaking of anti-Ramsey theorems was R. Rado, in Montreal, 1972, see [235]. Yet, that was slightly different. Babai [12] connects anti-Ramsey theorems to Sidon sets. This way Babai thinks that Anti-Ramsey theorems were first considered by Erdős and Turán, back in 1941 [138]. The topic I will shortly describe here is the following:

> Given a family $\mathcal{L}$ of forbidden graphs, (maximum) how many colors can one use to color the edges of K_n without having a copy of some $L \in \mathcal{L}$ colored with all different colors, i.e. colored in $e(L)$ colors. Denote the maximum by $\mathbf{AR}(n, \mathcal{L})$.

An edge-colored graph L will be called **Totally Multicolored** (TMC) if all its edges have distinct colors. It is clear that if we use $\binom{n}{2}$ colors, then we shall have a Totally Multicolored copy of any L with $v(L) \leq n$. If we use one color, then we shall have no Totally Multicolored L (except K_1 and K_2). So $\mathbf{AR}(n, \mathcal{L})$ is well defined.

To get some orientation, let us assign to each $\mathcal{L}$ a family $\mathcal{L}^*$ by taking for each $L \in \mathcal{L}$ and for each $e \in E(L)$ the graph $L - e$. These graphs form $\mathcal{L}^*$. Consider

Construction 14.1. *Take an extremal graph S_n for $\mathcal{L}^*$ and color its edges by (all distinct) $e(S_n)$ colors and color the edges of its complementary graph $\overline{G}_n$ by another color.*

This shows that

Claim 14.2. $\mathbf{AR}\,(n, \mathcal{L}) \geq \mathbf{ex}(n, \mathcal{L}^*) + 1.$

This estimate is often sharp:

Theorem 14.3 (Erdős, T. Sós and Simonovits [133]). *For $n > n_0(p)$,*

$$\mathbf{AR}\,(n, K_{p+1}) = \mathbf{ex}(n, K_p) + 1.$$

Montellano–Ballesteros and Neumann-Lara [226] removed the condition $n > n_0$ from the above theorem, by providing a quite elementary induction proof.

The cycle problem. As in the Erdős–Gallai Theorem, $\mathcal{C}_{\geq \ell}$ is the family of cycles of length $\geq \ell$. One of the unsolved problems of the field is to determine $\mathbf{AR}\,(n, \mathcal{C}_{\geq \ell})$. The next construction provides a lower bound and is conjectured to be sharp:

Construction 14.4 ([133]). *Partition n vertices into $\nu = \lceil \frac{n}{\ell-1} \rceil$ groups $U_1, \dots, U_\nu$ of $\ell - 1$ vertices (the last group may be smaller). All the pairs (x, y) within the groups are colored by an "own" color, i.e., by a color not used for other edges. The edges (x, y) $x \in C_i$, $y \in C_j$, for $j > i$ are colored by "i" (not used for the other edges).*

Conjecture 14.5. *The above coloring provides the (asymptotically?) best coloring for the $\mathbf{AR}\,(n, \mathcal{C}_{\geq \ell})$-problem.*

For $\ell = 3$ the problem is trivial, for $\ell = 4$ the conjecture was proved by Alon [5]. Recently Tao Jiang and D. West proved the conjecture for $\ell = 5, 6$ and provided some further estimates for $\ell > 6$, see [184]. The same result was independently proved, perhaps a few month later, by Ingo Schiermayer.

Related further literature: Tao Jiang [183].

An application of an Anti-Ramsey theorem. First Babai [12] then Babai and T. Sós [17] used anti-Ramsey theorems to prove estimates on Sidon subsets of abstract groups. This was also used to estimate the order of a Cayley graph containing a given graph.

Uniformity conditions. In Construction 14.1 all the colors of S_n were used only once, while the colors of the complementary graph had multiplicity $\geq cn^2$. This is very uneven. One can ask, what happens if we add an extra condition on the edge-color distribution, ruling out the very uneven coloring. Several related results can be found in some papers of Erdős and Tuza, see e.g. [139].

Trees of small diameter. I wish to mention here two new results. A. Bialostocki conjectured that if $\mathcal{T}_n(\ell)$ denotes the family of spanning trees of diameter ℓ, then

$$\mathbf{AR}\left(n, \mathcal{T}_n(4)\right) = \binom{n-2}{2} + 1 \quad \text{if} \quad n \geq 3.$$

B. Montágh proved this, moreover, he proved the stronger result:

Theorem 14.6 (Montágh [224]). *Let $\mathcal{T}_{n,k}$ be the family of trees of order n, diameter 3, with $n-k$ edges, where one of the degrees is at least $n-2k-2$. Then*

$$\mathbf{AR}\left(n, \mathcal{T}_{n,k}\right) = \binom{n-k-1}{2} + 1, \quad \textit{if} \quad n \geq \frac{11}{2}k + 7.$$

Improving an earlier result of Simonovits and T. Sós [257] Montágh proved that

Theorem 14.7 (Montágh, 2001 [225]). *If $n \geq k^2 + 7k$, then*

$$\mathbf{AR}\left(n, P_{n-k}\right) = \binom{n-k-1}{2} + 1.$$

15. Hypergraph Extremal Problems

Erdős wrote in [102]:

> As far as I know, the subject of hypergraphs was first mentioned by T. Gallai, in conversation with me in 1931. He remarked that hypergraphs should be studied as a generalization of graphs. The subject really came to life only with the work of Berge.

As we mentioned, we distinguish "intersection theorems for set systems" and "hypergraph extremal problems", but many people prefer to regard them together. Anyway, for hypergraphs, most of the extremal problems are unsolved or only partially solved. I recommend the paper of Füredi [151]. Chapter 6 of the Chung-Graham book [73] deals with open Erdős problems (in this field) and is a very good source to use. Also, for Intersection Theorems one of the basic sources could be, e.g., Peter Frankl's Chapter 24 in the Handbook of Combinatorics [145].

Hypergraph problems are much more involved than ordinary graph problems. Therefore, despite the fact that there are a few beautiful theorems in the field, in most cases we have only unsatisfactory results. A hypergraph is r-uniform if every edge has r vertices. Extremal problems extend to r-uniform hypergraphs, and the corresponding definitions of $\mathbf{EX}(n, \mathcal{L})$ and $\mathbf{ex}(n, \mathcal{L})$ are straightforward.

The oldest hypergraph extremal problems are due to Turán [280]. All hypergraphs are assumed to be uniform here.

Turán's Problem. For given r, p and n, how many edges can an r-uniform hypergraph H_n have without containing the complete hypergraph of order p?

This problem seems to be extremely difficult. Here we restrict ourselves to the simplest case of the corresponding conjecture, with $r = 3$ and $p = 4$.

Conjecture 15.1. *Let $T_{n,3}^{(3)}$ be the 3-uniform hypergraph whose n vertices are divided into three sets C_1, C_2 and C_3 (as nearly equal in size as possible), where a triple $\{x, y, z\}$ is an edge if no two of x, y, z are in the same set or if two belong to C_j and the third belongs to C_{j+1} (subscripts taken modulo 3).*

Turán's conjecture is that $T^{(3)}_{n,3}$ has the maximum number of edges among the 3-uniform hypergraphs of order n not containing the complete hypergraph of order 4.

Even this simplest case is unsolved. If the conjecture is true, then the extremal graph is not unique. In addition to there being some trivial variations of Turán's construction, a very nice construction was found by Brown [55] and then a more general one, by Kostochka [204].

Related literature: Katona, Nemetz, and Simonovits [188], Spencer [264], de Caen [63] Sidorenko [244].

15.1. Erdős hypergraph theorem

We now turn to the hypergraph version of the Kővári–T. Sós–Turán theorem. Let $K^{(r)}_r(m)$ denote the complete r-partite r-uniform hypergraph with m vertices in each of its classes, where the edges are the r-tuples with one vertex from each class. The problem is to determine how many r-tuples can H_n have without containing $K^{(r)}_r(m)$. Bounds for this number were found by Erdős[33] [85]:

Theorem 15.2. *There exists an n_0 such that for $n > n_0$*

$$n^{r-rm^{1-r}} \le \mathbf{ex}\big(n, K^{(r)}_r(m)\big) \le n^{r-cm^{1-r}}.$$

The proof of the upper bound of this theorem is not much more difficult than that of the Kővári–T. Sós–Turán theorem: as a matter of fact, it can be reduced to iterating the argument of the proof of Theorem 4.1. The lower bound is obtained by the method of random hypergraphs.

The Problem of Jumping Constants. *Theorem* 15.2 *has the following consequence:*

Corollary 15.3. *Let $\varepsilon > 0$, and let (S_n) be a sequence of r-uniform hypergraphs such that $e(S_n) > \varepsilon\binom{n}{r}$. Then S_n contains a subhypergraph H_m with $e(H_m) > \frac{r!}{r^r}\binom{m}{r}$ and $m \to \infty$ as $n \to \infty$.*

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[33] As Füredi pointed out to me, formally the lower bound was incorrect in Erdős's theorem: the exponent in the lower bound was $r - cm^{1-r}$. Since no proof was given, it is difficult to decide today what Erdős really meant there. For us the upper bound is the really important one.

This means that if the edge-density of S_n is positive, then, for some appropriately chosen subgraphs, this density must jump up (independently of ε) to $\frac{r!}{r^r}$. The problem of the jumping constants can be formulated in its most special form (with $r = 3$) as follows:

Problem 15.4. *Does there exist a constant $c > 0$ (independent of ε) such that if (S_n) is a sequence of 3-uniform hypergraphs for which $e(S_n) > \left(\frac{1}{27} + \varepsilon\right) n^3$, then there exists a sequence of subgraphs $H_m \subseteq S_n$ (where $m \to \infty$ as $n \to \infty$), for which $e(H_m) > \left(\frac{1}{27} + c\right) m^3$?*

Define the edge-density of a sequence of r-uniform hypergraphs S_n by

$$\limsup \frac{e(S_n)}{\binom{n}{r}}.$$

The general problem is

Problem 15.5 (Erdős). *Characterize those constants c for which there exists an $f(c) > c$ such that, whenever the edge-density of (S_n) is larger than c, then there is a sequence of subgraphs $H_m \subseteq S_n$ $(m \to \infty)$ for which the edge-density of H_m is at least $f(c)$.*

These are the "jumping constants". By the Erdős–Stone theorem, every c is a "jumping constant" for $r = 2$. Clearly, $c = 0$ is also a jumping constant, for any r. Frankl and Rödl showed that there are infinitely many extremal densities which are *not* jumping constants (for 3-uniform hypergraphs) [146].

15.2. Brown–Erdős–T. Sós theory

W. G. Brown, Erdős and T. Sós wrote two papers [56], [57]. The first paper consisted (in some sense) of two separate parts. Here we regard the density problems:

> Given the parameters r, k, and ℓ, what is the maximum number of hyperedges an r-uniform hypergraph can have without having a k-vertex subhypergraph of at least ℓ hyperedges.

Let $\mathcal{L}_{k,\ell}$ denote the family of r-uniform hypergraphs with k vertices and ℓ edges. Brown, Erdős and T. Sós [56], [57] began investigating the function $f(n, k, \ell) = \mathbf{ex}(n, \mathcal{L}_{k,\ell})$. (The idea of investigating $f(n, k, \ell)$ goes back most probably to the Smolenice paper [86], see Section 6.4.)

This "frame" includes many interesting problems: the $(4,4)$ problem is Turán's hypergraph problem, the $(4,2)$ problem is just the problem of Steiner triple systems. Also, the $(4,3)$ problem is among the famous hypergraph extremal problems, see Section 5 of [259].

The problem of finding good lower and upper bounds is fairly simple for some pairs k,t and extremely difficult for others. Brown, Erdős and T. Sós have solved many cases of this general frame. The first real difficulty which they encountered was the case $r = \ell = 3$, $k = 6$. Although they could not settle this problem, Ruzsa and Szemerédi [241] later found an astonishing result.

Let $r_k(n)$ denote the maximum number of integers in $[1,n]$ containing no arithmetic progression of length k. Szemerédi's famous theorem [272] asserts that $r_k(n) = o(n)$. Further, it is known (see Behrend, [21], Heath-Brown [175], Szemerédi [274], Bourgain [51]) that

$$n^{1-c/\sqrt{\log n}} \leq r_3(n) \leq c' \frac{n}{(\log n)^{\frac{1}{2}+\varepsilon}}$$

One can easily see that $f(n,6,3) < cn^2$.

Theorem 15.6 (Ruzsa–Szemerédi). *There exists a $c > 0$ such that*

$$cn \cdot r_3(n) \leq f(n,6,3) = o(n^2).$$

Clearly, the Ruzsa–Szemerédi theorem implies $r_3(n) \leq f(n,6,3)/n = o(n)$.

One reason, why this result is so surprising is that it implies the nonexistence of an α such that (for some $c_1 > 0$)

$$c_1 n^\alpha \leq f(n,6,3) \leq c_2 n^\alpha.$$

Though we cannot prove it, we are convinced that for ordinary graphs the situation is completely different (see Conjecture 4.29).

Frankl and Rödl have tried to extend this and prove $r_4(n) = o(n)$ using an appropriate extremal problem which could be solved by an appropriate version of the Szemerédi Regularity Lemma, and they have announced that they succeeded in this for $r_4(n)$. Until now nobody knows if the general case can be solved this way.

We close this part with an unpublished result of Füredi:

Theorem 15.7. $\mathbf{ex}(n, \mathcal{L}_{5,3}) = \frac{n^2}{5} + o(n^2)$.

Applications of the Ruzsa–Szemerédi theorem. The Ruzsa–Szemerédi theorem has several applications in graph theory.

(a) One of the most interesting applications is

Theorem 15.8 (Füredi [154]). *If G_n is a graph of diameter 2 but deleting any edge of G_n the diameter increases to 3, then $e(G_n) \le e(T_{n,2})$ if $n > n_0$, with equality holding if and only if $G_n = T_{n,2}$.*

(b) Another application is when Gyárfás first generalized it as follows:

Theorem 15.9 (Gyárfás [167]). *If G_n is a graph which can be edge-colored in cn colors so that each color defines a subgraph of maximum degree d, then $e(G_n) = o_d(n^2)$.*

The case $d = 1$ is equivalent with the $(6,3)$ problem. Gyárfás used this result as a lemma to prove a conjecture of Füredi and Kahn [157] on the dimension of lattices.

Burr, Erdős, Graham and T. Sós [61, 62] started investigating the following "anti-Ramsey" problem:

> Given a sample graph L, and two integers n and e, what is the minimum number of colors, $k = k(n, e, L)$ for which there exists a G_n with $e = e(G_n)$ edges that we can edge-color in k colors so that every $L \subseteq G_n$ is Totally Multicolored (i.e. all the edges of L have distinct colors).

Clearly, this is interesting only for $e > \mathbf{ex}(n, L)$. Burr, Erdős, Graham and T. Sós [61] gave various upper and lower bounds on the minimum of $k(G, L)$ for fixed n and e. One important and striking fact was that for $L = P_4$ to solve the problem is equivalent with proving the Ruzsa–Szemerédi theorem. Several related results can be found in [61, 62] and [132].

Related further literature: Simonovits [1].

15.2.1. Hypergraphs and color-critical graphs. A graph is k-edge-color-critical if it is k-chromatic and deleting any edge of it we get a $k-1$-chromatic graph. The 3-critical graphs are uninteresting: they are just the odd cycles. Many questions were formulated by Dirac and Erdős about critical graphs. Erdős asked, how many edges can a k-critical graph have. Dirac, in return, showed his 6-critical graph $G_n := C_{2\ell+1} \otimes C_{2\ell+1}$ with $\left\lfloor \frac{n^2}{4} \right\rfloor + n$ edges. Gallai also had several interesting results and conjectures on color-critical graphs. One of his conjectures asserted that color-critical graphs cannot have too many independent vertices. I have invented a method which, if applied to the Dirac construction, gave a 6-critical graph with $n - o(n)$ vertices of degree 5, completely disproving Gallai's conjecture, but only for 6-critical graphs. Then I decided not to publish this result since I learned from Erdős that Brown and Moon [58] (roughly the same time) found a much more constructive and much stronger counter-example to Gallai's conjecture.

Next fall Bjarne Toft visited Budapest and I learned of his very simple and very nice construction of 4-critical graphs with many edges [276]. We had some discussion with Toft and then independently, but in a very similar way obtained a graph G_n which was 4-critical and its minimum degree was $c\sqrt[3]{n}$ [277], [250]. Further, I proved (using a hypergraph extremal problem) that a 4-critical graph G_n cannot have more than $n - O(n^{2/5})$ independent vertices.

Let $\mathcal{L}_{TRI}$ denote the family of 3-uniform hypergraphs obtained by taking an arbitrary triangulation of the sphere in $\mathbb{R}^3$.

Theorem 15.10 (Brown–Erdős–T. Sós [56]).

$$c_1 n^{5/2} \le \mathbf{ex}(n, \mathcal{L}_{\mathrm{TRI}}) \le c_2 n^{5/2}.$$

Let $\mathcal{L}_{\text{r-CONE}}$ be the family of 3-uniform all the hypergraphs H_{r+k} obtained by taking a cycle $x_1 x_2 \dots x_k x_1$ and r new vertices $y_1, \dots, y_r$ and taking the triplets $x_i x_{i+1} y_j$ for $j = 1, \dots, r$ and $i = 1 \dots, k$ (where $x_{k+1} := x_1$).

Theorem 15.11 (Simonovits [250, 251]). $\mathbf{ex}(n, \mathcal{L}_{\text{r-CONE}}) = O\big(n^{3-1/r}\big)$ *and for* $r = 2, 3$ *the upper bound is sharp.*

I arrived at this result investigating color-critical graphs. I proved it independently from Brown, Erdős and T. Sós. For $r = 2$ their upper bound is formally weaker, but actually they prove exactly what I proved and their lower bound is stronger. For $r = 3$, I generalized Brown's construction [54]: I used a "modified" finite geometric construction. L. Lovász [210] improved my estimate on the Gallai problem, by taking a wider class of forbidden graphs: Let $\mathcal{L}_{\text{SPERNER}}$ be the family of all the 3-uniform hypergraphs in which every pair of vertices is in an *even* number of 3-edges.

Theorem 15.12 (Lovász [210]). $\mathbf{ex}(n, \mathcal{L}_{\text{SPERNER}}) = O(n^2)$.

Using my method and this stronger extremal result, Lovász proved

Theorem 15.13. *Let $\alpha_k(n)$ denote the maximum number of independent vertices in a k-critical graph on n vertices. Then*

$$n - 2kn^{1/(k-2)} \le \alpha_k(n) \le n - (k/6)n^{1/(k-2)}.$$

The importance of this paper lies (among others) in that this is where Lovász started using linear algebra methods to graph problems.

Related literature: Bollobás, Chapter 5 of [29], Lovász [212].

15.3. Some solved hypergraph extremal problems

(a) G.O.H. Katona conjectured that if a 3-uniform hypergraph has $3n$ vertices and n^3+1 triples, then there are two triples whose symmetric difference is contained in a third triple. The 3-partite hypergraph $K_3^{(3)}(n, n, n)$ showed that if true, the conjecture is sharp. Bollobás proved this conjecture [26][34], and conjectured some generalizations which were proved for 4-uniform hypergraphs by Sidorenko [243] and some related estimates were also obtained for the general case by D. de Caen [64].

(b) Color-critical graphs play an important role in the theory of ordinary extremal graphs: these are where the extremal graphs are the simplest: $T_{n,p}$. For hypergraphs perhaps the Fano plane is one of the simplest color-critical graphs. One corresponding conjecture of V. T. Sós was:

Conjecture 15.14. *Let F_7 be the 3-uniform hypergraph defined by the 7 triples of the Fano Plane. Is it true that (for n large) one extremal graph for F_7 is the graph obtained by splitting n vertices into two parts of order $\approx \frac{1}{2}n$ and taking all the triplets having vertices in both classes.*

The conjecture has recently been proved by de Caen and Füredi, [66], applying some multigraph results of Füredi and Kündgen [158]. For more details, see our multigraph survey with W. G. Brown [60].

Related literature: Mubayi and Rödl [230].

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[34] Bollobás mentioned that Erdős drew his attention to this problem

16. Complete?

Is this survey sufficient to get a good picture about these fields? By no ways. These few pages are not enough to describe such a huge area. Among many other topics, I completely left out everything on the list-chromatic number [123], Erdős results on Tournaments, the Erdős–Goodman–Pósa [111] results and the Hajnal-Szemerédi theorem on the Erdős conjecture [171]. I also left out here many results about covering a graph with given type subgraphs, results of Erdős and T. Sós and others on "unavoidable structures" ... and many further interesting and important fields.

Many applications, motivations can be found (as I stated) in [60], [259], and also in my notes to Turán's collected papers [284].

But let me stop here. ■

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